2020

Lyapunov Stability Criterion and Stability Analysis using this Criterion

Introduction

The stability of linear system can be determined by asymptotically stability criterion and Bounded-Input-Bounded-Output (BIBO) stability criterion. But Lyapunov stability criterion is basically useful for checking the stability of nonlinear systems. Actually, stability of both linear and non-linear systems can be determined using Lyapunov method.

Lyapunov stability criterion was developed by A. M. Lyapunov in 1892 and because of his name this criterion is known as Lyapunov stability criterion. This criterion is based on the concept of energy if the total energy of a system is dissipated then the system is always stable.

Equilibrium State

A state of an autonomous system is called an **equilibrium state** if the system will stay at that point after starting from that point in the absence of the forcing input.
If a system is described by $\dot{X} = f(X)$, then at equilibrium state (X_e),

$$\dot{X} = f(X_e) = 0.$$

Stability in the sense of Lyapunov

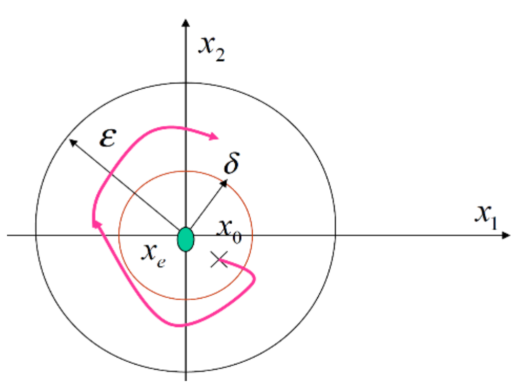
Consider an autonomous system described by

$$\dot{X} = f(X)$$

The above system is said to be stable in the sense of Lyapunov about an equilibrium point X_e if for every $\varepsilon > 0$,there exist $\delta(\varepsilon) > 0$ such that $\|x_0 - x_e\| < \delta$ and $\|x(t) - x_e\| < \varepsilon$ for $\forall t \geq 0$.

Here $\|.\|$ represents the Euclidean norm. The $\|X\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$, which represents a hyper-spherical region.

A dynamic system is *Lyapunov stable* about an equilibrium point if state trajectories are confined to a bounded region whenever the initial condition is chosen sufficiently close to equilibrium point.

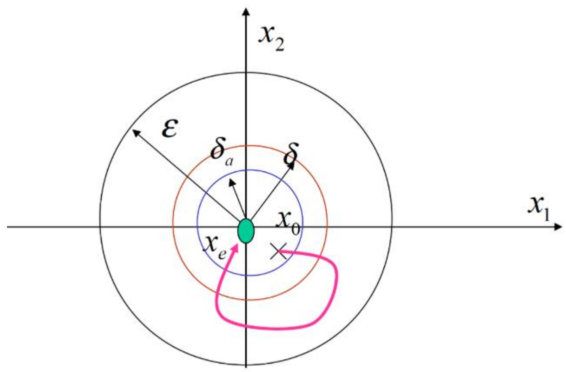


Asymptotically Stability Theorem

An equilibrium state X_e of an autonomous system is asymptotically stable if

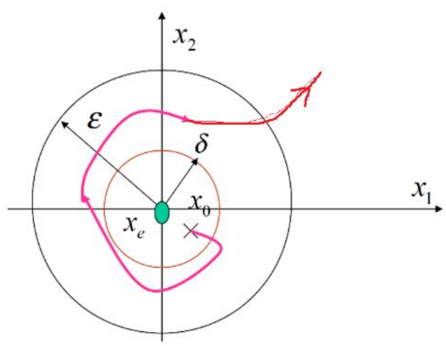
- it is stable and $\delta_a < \delta$
- exist if $\|x_0 - x_e\| < \delta_a$ and $\|x(t) - x_e\| \to 0$ as $t \to \infty$.

For asymptotically stable, state trajectories converge to the equilibrium point.



Instability Theorem

A dynamic system is unstable about an equilibrium point if state trajectories leave a bounded region whenever the initial condition is chosen sufficiently close to equilibrium point.



Local Stability (stability in-the-small) and Global stability (Stability in-the-large)

The stability in the sense of Lyapunov is applicable in a small region as in this definition, the region considered around the equilibrium point is very small. So, this type of stability is known as local stability or stable in-the-small region.

If the region considered is large i.e. if entire state-space is considered then the stability is known as stable in-the -large or global stability.

Definiteness of scalar function

Positive definite function:

The function $f(x)$ is *positive definite* if $f(0) = 0$ and $f(x) > 0$ for all $x \in R^n$.
Example: $f(x_1, x_2) = x_1^2 + x_2^2$.

Positive semi-definite Function:

The function $f(x)$ is *positive semidefinite* if $f(0) = 0$ and $f(x) \geq 0$ for all $x \in R^n$.
Example: $f(x_1, x_2) = (x_1 + x_2)^2$.

Negative definite function:

If the function $-f(x)$ is *positive definite* then function is called negative definite function.
Example: $f(x_1, x_2) = -(x_1^2 + x_2^2)$.

Negative semi-definite function:

If the function $-f(x)$ is *positive semidefinite* then function is called negative semidefinite function.
Example: $f(x_1, x_2) = -(x_1 + x_2)^2$

Indefinite function:
A function is not definite or semidefinite in either sense is defined to be **indefinite function.** For some values of *x*, *f(x)* is positive and for some values *x*, it is negative.
Example: $f(x_1, x_2) = x_1 x_2 + x_2^2$.

Sign definiteness of quadratic function

Consider a continuous function *V* (*x*) that can be expressed as

$$V(x) = x^T P x$$

where $P \in R^{n \times n}$ is a symmetric matrix.

If the matrix P is positive definite, then the corresponding quadratic form $V(x)$is also will be positive definite.
The sign of matrix P can be determined using *Sylvester's criterion* and also form the eigenvalues of the matrix P.

Sylvester's criterion:
A symmetric matrix *P* is positive definite if and only if all its principle minors are positive.

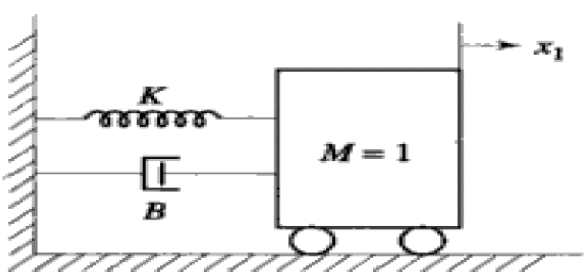
Consider *P* is a 3×3 symmetric matrix and $P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12} & P_{22} & P_{23} \\ P_{13} & P_{23} & P_{33} \end{bmatrix}$

If $P_{11} > 0$; $\begin{vmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{vmatrix} > 0$ and $\begin{vmatrix} P_{11} & P_{12} & P_{13} \\ P_{12} & P_{22} & P_{23} \\ P_{13} & P_{23} & P_{33} \end{vmatrix} > 0$, then *P* will be positive definite.

Alternatively, the sign definiteness of *P* can be determined from the eigen values of *P* also. The matrix *P* will be positive definite, if and only if all the eigenvalues of *P* are positive. If eigenvalues are positive or zero, then *P* will be positive semidefinite.

Lyapunov stability criterion (Direct method)

In this approach, stability of the system can be determined without solving the differential equation. So, this approach is known as direct method. The direct method of Lyapunov stability criterion is based upon the concept of energy and the relation of stored energy with system stability.



Consider a spring-mass-damper system as shown in Fig. above. Here the dynamics of the system is given by-

$$\ddot{x} + D\dot{x} + Kx = 0$$

Where x is the linear displacement of the body, M is the mass of the body and here M=1, D is the damping co-efficient and K is the spring constant.

Here state variables are-

$$x = x_1 \; ; \dot{x} = x_2$$

So the equations are-

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -Dx_2 - Kx_1$$

For this system, the energy storage elements are mass and the spring. So the internal energy stored by the system-

$$V(X) = \frac{1}{2}x_2^2 + \frac{1}{2}Kx_1^2$$

The rate of change of energy is given by-

$$\dot{V}(X) = x_2\dot{x}_2 + Kx_1\dot{x}_1$$

$$= x_2(-Dx_2 - Kx_1) + Kx_1x_2$$

$$= -Dx_2^2$$

From the above equation, it can be observed that the rate of change of internal stored energy is negative. So here internal stored energy decreases. So, the system is here a stable system.

So, an energy function is associated with the direct method of the Lyapunov stability criterion and that energy function should be positive definite. This energy function can be determined easily if the physical system is known to us. But it is very difficult to determine energy function from the mathematical equations of the system if the system components are not known to us. In that case fictious energy function is considered. This fictious energy function is known as **Lyapunov function** and the Lyapunov function should be positive definite function.

Lyapunov's stability theorem (direct method)

Consider an autonomous system described by

$$\dot{X} = f(X) \tag{1}$$

Suppose that there exists a scalar function $V(x)$ which for some real number $\varepsilon > 0$, satisfies the following properties for all *x* in the region $\|x\| \leq \varepsilon$:

- *V*(*x*) is positive definite function i.e. $V(x) > 0$; $x \neq 0$ and $V(0) = 0$.
- *V*(*x*) has continuous partial derivatives with respect to all components of *x*.

The equilibrium state $x_e = 0$ of the system given by equation (1) is

- asymptotically stable if $\dot{V}(x) < 0$, $x \neq 0$, i.e., $\dot{V}(x)$ is a negative definite function.
- asymptotically stable in-the-large if $\dot{V}(x) < 0$, $x \neq 0$ and in addition $V(x) \to \infty$ as $\|x\| \to \infty$.

Lyapunov's instability theorem (direct method)

Consider an autonomous system described by

$$\dot{X} = f(X) \tag{2}$$

Suppose that there exists a scalar function $V(x)$ which for some real number $\varepsilon > 0$, satisfies the following properties for all x in the region $\|x\| \leq \varepsilon$:

- $V(x)$ is positive definite function i.e. $V(x) > 0$; $x \neq 0$ and $V(0) = 0$.
- $V(x)$ has continuous partial derivatives with respect to all components of x.

The equilibrium state $x_e = 0$ of the system given by equation (2) is unstable if $\dot{V}(x) > 0$, $x \neq 0$, i.e., $\dot{V}(x)$ is a positive definite function.

Lyapunov function for linear system

Using Lyapunov's direct method, stability of both linear and non-linear system can be determined. Consider a linear autonomous system described by

$$\dot{x} = Ax \tag{3}$$

where A is $n \times n$ real constant matrix.

The linear system described by equ. (3) is globally asymptotically stable at the origin if and only if, for any given symmetric positive definite matrix Q, there exists a symmetric positive definite matrix P that satisfies the matrix equation

$$A^T P + PA = -Q$$

Proof: Consider Lyapunov function $V(x) = x^T P x$

where $V(x)$ is positive definite function.

Then $\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x}$

$$= x^T A^T P x + x^T P A x$$

$$= x^T (A^T P + PA) x$$

$$= -x^T Q x$$

$\therefore A^T P + PA = -Q$

The above equation is known as Lyapunov equation for linear system.

As Q is positive definite, $\dot{V}(x)$ will be negative definite. So, the system will be stable.

Norm of x is defined as $\|x\| = (x^T P x)^{1/2}$

So $V(x) = \|x\|^2$

$V(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$

So, the system is here globally asymptotically stable at the origin.

Note:

- The system will be asymptotically stable if for positive definite matrix Q, the solution of Lyapunov equation P is also becomes positive definite.
- The matrix P is here symmetric matrix.
- For convenience the matrix Q is often chosen as identity matrix.

Krasovskii's method for nonlinear system stability analysis

Consider a nonlinear system described by the dynamic equation

$$\dot{X} = f(X)$$

The Lyapunov function for the system is

$$V(X) = f^T(X)f(X)$$

Here V(X) is positive definite function.

Then $\dot{V}(x) = \dot{f}^T f + f^T \dot{f}$

$$= \dot{X}^T (\frac{\partial f}{\partial X})^T f + f^T \frac{\partial f}{\partial X} f$$ [here $\frac{\partial f}{\partial X}$ is the Jacobian matrix]

$$= f^T \left(\frac{\partial f}{\partial X}^T + \frac{\partial f}{\partial X}\right) f$$

$$= f^T F f$$

Here if F is negative definite, $\dot{V}(x)$ is also will be negative definite and the nonlinear system will be stable.

Books:

- J. Nagrath, M. Gopal, Control System Engineering, fifth edition, 2007.
- K. R. Verma, Modern Control Theory, CBS Publishers and Distributors Pvt Ltd, 2018.