Date - 22.01.20

Rigid body, Fluide body, Reversed productive force.

$$F_{external} + (-R) = 0.$$

$$R = fa$$

Reversed effective force. $\int_m dm\, f_m$

$$= f_m \int_m dm$$

$$= mf.$$

Velocity = $\dfrac{\text{Rate of change of displacement along } \overrightarrow{OP}}{\text{Change of time}}$

$\frac{x - 0}{t - 0} = \frac{x}{t}$

$$\lim_{\Delta t \to 0} \frac{x + \Delta x - x}{t + \Delta t - t}$$

$$\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}.$$

central gravity.

$$M = \sum dm = \int dm.$$

$$M \frac{d^2x}{dt^2} = Fe = F_H + (-F).$$

Date - 28.01.20

at the initial time when time counting.

$t = 0$, $T = 10.30$.,

$T^* = t - 10.30 = 0.$

Starting point

Reversed effective force $= mf.. = m \frac{d^2x}{dt^2}$.

$F \propto t$, $F \propto x$.

$F = \mu x^2$ (helping force) $R = \frac{v^2}{x}$. energy redu

$$m\frac{d^2x}{dt^2} = F_1 + (-F_2) = \alpha x^2 - \frac{\gamma}{x}.$$

or $$\frac{d^2x}{dt^2} = \frac{\alpha}{m}x^2\left(\frac{\gamma}{m}\right)$$

$$\frac{d^2x}{dt^2} = K_1x^2 - K_2/x \qquad K_1 = \frac{\alpha}{m},\ K_2 = \frac{\gamma}{m}$$

$$x(t) = 5t - \frac{t}{2}\ 11\ t^2 \qquad \text{what's } x(t) = ?$$

$$v = \frac{dx}{dt} = ?$$

$$x(t) - x(0) = \pm ft.$$

$$x(t) = v = x(0) \pm ft.$$

$$v = u \pm ft.$$

$$\frac{dx}{dt} = u + ft.$$

$$\int_0^x dx = \int (u + ft)dt.$$

$$\Rightarrow x(t) = u \pm \frac{1}{2}ft^2.$$

$$\left[x = \frac{1}{2}vt\right]$$

Q Ex-II Pg-28

1.

$$v(0) = \frac{dx}{dt}\bigg|_{t=0} = u.$$

Force $= m \times acc^n$.

$= m(-\omega^2 x)$

$= -m(\mu \times x)$

$= -m\mu x$

The eq^n of motion of the partical object to the given force is $\cancel{m}\frac{d^2x}{dt^2} = -\cancel{m}\mu x$

$$f = \frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{dx}{dt}\right) = \frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt}$$

$$= v\frac{dv}{dx}.$$

$\Rightarrow v\frac{dv}{dx} = -\mu x$

So, $2v\,dv = -\mu\,2x\,dx$ so integration $v^2 = -\mu x^2 + c$

at $x = 0$, $v = u \Rightarrow c = v^2$

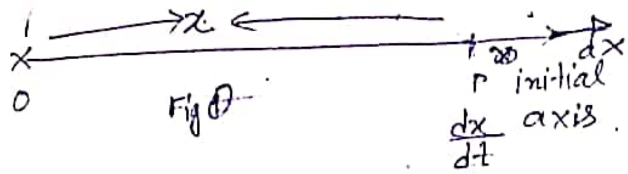
Date. - 04.02.20

② Motion in a straight line:

imaginary plane or organ plane.

⑪ Problem:

$F = mf$ variable accelaration. $v = 0$



at any time

$$\frac{dx}{dt} = \mu \frac{\sqrt{a-x}}{x}.$$

Prove that, $F (= m\ddot{x}) \propto \frac{1}{x^2}$.

$$= m \left(\frac{d}{dx}\left(\frac{dx}{dt}\right)\right)$$

soln
⇒ Let, p be the position of the partical an any time t s.t. $op = x$ as seen in Fig: 1.

Hence, the velocity v (say) at any time, then we have,

$$v = \frac{dx}{dt} = \mu\sqrt{\frac{a-x}{x}} \quad \text{—①}$$

differentiating ① we have.

$$\frac{dv}{dt} = \mu \frac{1}{2}\left(\frac{a-x}{x}\right)^{-1/2} \cdot \frac{d}{dx}\left(\frac{a-x}{x}\right)$$

$$= \frac{\mu}{2} \cdot \frac{1}{\sqrt{\frac{a-x}{x}}} \cdot \frac{-(a-x) + x(-1)}{x^2}$$

$$= \frac{\mu}{2}\sqrt{\frac{x}{a-x}} \left(\frac{-a}{x^2}\right) \quad \text{—②}$$

Now the acceleration of the partical at P is .

$$\frac{d^2x}{dt^2} = \cancel{\frac{dx}{dx}}\,\frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dx}\left(\frac{dx}{dt}\right)\left(\frac{dx}{dt}\right)$$

$$= \frac{dv}{dt}\, v .$$

~~so~~

$$= \frac{\mu}{2}\sqrt{\frac{x}{a-x}}\left(-\frac{a}{x^2}\right)\cdot \mu\sqrt{\frac{a-x}{x}} .$$

$$= \frac{\mu^2}{2}\left(-\frac{a}{x^2}\right) .$$

$$= \left(-\frac{a\mu^2}{2}\right)\cdot\frac{1}{x^2}$$

$$f_{at\ p.} = k\cdot\frac{1}{x^2} \quad \text{where } k = -\frac{\mu^2 a}{2} .$$

$\therefore$ Acceleration at any time $f \propto \frac{1}{x^2}$.

$\boxed{\text{Barik: } 18, 19, 21, 23, 24}$

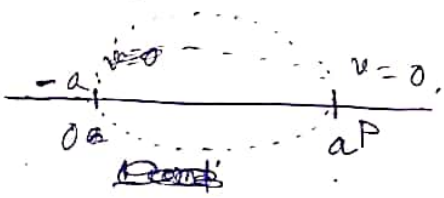
19

Soln:-

⇒

$x = a,\ v = 0$

One complete cycle = occilation.



⇒ Let,

It is given in the ~~given~~ question accn of the particle ~~given in the questi~~

$\frac{d^2x}{dt^2}$ / $v\cdot\frac{dv}{dx}$ any time t .

Therefore,

$$v \cdot \frac{dv}{dx} = \frac{\mu}{x^2} - \frac{\lambda}{x^3} \quad \text{where, at } x = a, \ \frac{dx}{dt} = v = 0 .$$

$$\Rightarrow \int 2v \frac{dv}{dx} = \int \frac{2\mu}{x^2}\, dx - \int \frac{2\lambda}{x^3}\, dx + A$$, where A is an arbitrary constant.

$$\Rightarrow v^2 = 2\mu\left(-\frac{1}{x}\right) + \lambda \frac{1}{x^2} + A$$

at $x = a$, $v = 0$

$$\Rightarrow 0 = 2\mu\left(-\frac{1}{a}\right) + \lambda \frac{1}{a^2} + A$$

$$\Rightarrow A = \frac{-\lambda + 2\mu a}{a^2}$$

Thus,

$$v^2 = \frac{2a\mu - \lambda}{a^2} + \frac{\lambda}{x^2} - \frac{2\mu}{x}$$

$$\Rightarrow \frac{a2\mu \cancel{+\lambda} \cancel{+\lambda} - 2\mu x}{\cancel{x^2 a^2}} = 0$$

$$\Rightarrow \frac{2\mu \cancel{(a - x)}}{x^2 \cancel{a^2}} = 0 .$$

$$\Rightarrow \frac{2a\mu x^2 - \lambda x^2 + \lambda a^2 - 2\mu x a^2}{x^2 a^2}$$

$$\Rightarrow \frac{2\mu a x(x - a) + \lambda(a - x)(a + x)}{x^2 a^2}$$

$$= \frac{(a - x)\left(-2\mu a x + \lambda(a + x)\right)}{x^2 a^2}$$

$$= \frac{(a - x)}{x^2 a^2} (2\mu a - \lambda)\left(x - \frac{\lambda a}{2\mu a - \lambda}\right)$$

Let, $\cancel{x = \frac{\lambda a}{2\mu a - \lambda}}$

The other ~~particle~~ position (except $x = a$) at which the particle will be at rest given by the sol^n of the eq^n $v = 0$.

let, $x = \frac{\lambda a}{(2\mu a - \lambda)}$

$$v = \frac{dx}{dt} = \sqrt{\frac{(2a\mu - \lambda)(a - x)\left(x - \frac{\lambda a}{2\mu a - \lambda}\right)}{xa}}$$

$$v = \sqrt{\frac{(x-a)(x-b)}{x}} \sqrt{\frac{\lambda}{ab}} \qquad b = \frac{\lambda a}{2\mu a - \lambda}$$

let, T be the time of oscillation, then T is given by -

$$T = 2\int_{x=a}^{x=b} dt$$

$$= 2\sqrt{\frac{ab}{\lambda}} \int_{x=a}^{x=b} \frac{x}{\sqrt{(x-a)(x-b)}}\, dx.$$

$(x-a)^2 = u^2$

$dx = 2u\, du.$

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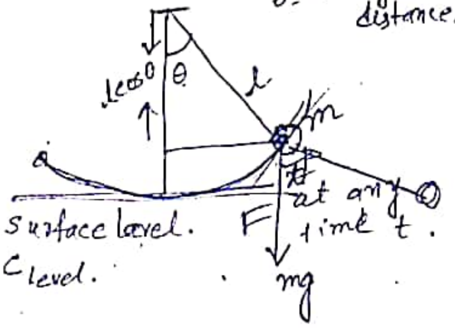
Simple harmonic motion,

SHM

$\theta = x =$ angular distance.



Force along (the tangential direction).

cross radial, is

$mg(\cos(\frac{\pi}{2}+\theta))$

$= mg \sin\theta$.

$l =$ constant

$\frac{dl}{dt} = 0$.

$\frac{d^2l}{dt^2} = 0$.

$\sin x \approx x$

$\cos x \approx 1$.

$r = l$.

$$F = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$$

$$F_r = \frac{1}{r}\frac{d}{dt}(r^2\theta)$$

$$= \frac{1}{l}\frac{d}{dt}(l\theta).$$

∴ the eqn of motion.

$$ml\frac{d^2x}{dt^2} = -mg\sin x$$

$$\Rightarrow \frac{d^2x}{dt^2} = -g/l\, x.$$, since x is small and ~~mass~~

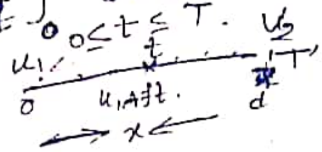
$$T = \frac{2\pi}{w}$$

$$\Rightarrow \frac{d^2x}{dt^2} = -w^2x, \quad w = \sqrt{\frac{g}{l}}.$$

5/2/2020

(b)

space avarage velocity $V_{avarage} = \frac{1}{d}\int_0^d v\,dx$. $0 \le t \le T$.



time avarage velocity $= \frac{1}{T}\int_0^T (u_1 + ft)\,dt$

$$= \frac{1}{T}\left[u_1 t + f\frac{t^2}{2}\right]_0^T$$

$$= \frac{1}{T}\left(u_1 T + \frac{1}{2} f T^2\right)$$

$$= u_1 + \frac{1}{2} f T.$$

final velocity,

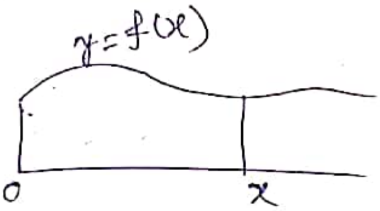
$u_2 = u_1 + fT$ $\qquad = u_1 + \frac{1}{2}(u_2 - u_1)$

$\Rightarrow fT = u_2 - u_1,$ $\qquad = \frac{u_1 + u_2}{2}$.

space avarage ~~velocity~~ $= \frac{1}{d}\int_0^d f(x)\,dx$.

$$\frac{1}{d}\int_0^d v\,dx$$

$$= \frac{1}{d}\int_0^d \sqrt{u_1^2 + 2fx}\,dx$$



$$= \frac{1}{x}\int_0^x v\,\frac{dx}{dt}\,dt.$$

~~space avarage velocity~~

$$V_{space} = \frac{\int_0^x v\,dx}{\int_0^x dx}$$

$$= \frac{1}{x}\int_0^x v^2\,dt.$$

$$= \frac{1}{xf}\int_{u_1}^{u_2} v^2\,dv$$

$v^2 = u_1^2 + 2fs$. $, 0 \le s \le x$.

or, $u_2^2 = u_1^2 + 2fx$

$\Rightarrow x = \frac{u_2^2 - u_1^2}{2f}$

$$= \frac{2}{u_2^2 - u_1^2}\cdot\left[\frac{v^3}{3}\right]_{u_1}^{u_2}$$

$\Rightarrow$

$v = u_1 + ft$

$$= \frac{2(u}{(u_2 + u_1)(u_2 - u_1)}$$

$dv = d(u_1 + ft)$

$= du_1 + d(ft)$

$= 0 + f\,dt.$

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(4)

$$\frac{d^2x}{dt^2} = f - kt^2 \text{, wh}$$

$$d\left(\frac{dx}{dt}\right) = (f - kt^2)\,dt.$$

on integration, we have,

$$\frac{dx}{dt} = ft - \frac{k}{3}t^3 + A,$$ where A = constant.

Inital, $t = 0$.

$$\frac{dx}{dt} = 0.$$

Thus we have $A = 0$

velocity at any time, $= ft - \frac{k}{3}t^3$.

$$v_{max} = \max\left\{ft - \frac{kt^3}{3}\right\}.$$

$$= ft - \frac{kt^3}{3},$$ where $acc^n = 0$

$f - kt^2 = 0$

$\Rightarrow t^2 = \frac{f}{k}$

$\Rightarrow t = \sqrt{\frac{f}{k}}$

$$= \left(f - \frac{k}{3}\frac{f}{k}\right)t$$

$a = f - kt^2$

$\frac{da}{dt} = -2kt < 0$

$$\cancel{= \frac{2}{3}\sqrt{\frac{f}{k}}}.$$

$$= \left(f - \frac{k}{3}\cdot t^2\right)t$$

$$= \left(f - \frac{k}{3}\cdot\frac{f}{k}\right)\sqrt{\frac{f}{k}}$$

$$= \left(\frac{2}{3}f\sqrt{\frac{f}{k}}\right) = \frac{2}{3}\sqrt{\frac{f^3}{k}}$$

force of attraction,

force of attraction.

force of repution

$x<,$ $t>0$, then, $\frac{dx}{dt} = +ve$, increasing f^n.

Simple Hermonic motion: (SHM)

The most basic example of SHM is give,

$$\frac{d^2x}{dt^2} = -w^2 x. \quad , w = \text{fequency}.$$

$$\frac{d^2x}{dt^2} = -\frac{g}{l}\sin x.$$

$$\sin x \approx x.$$

SHM is subject to three fundamental force.

(i) Controlling force.

$$F_c = m w^2 x.$$

(ii) Damped force. / Resisting force.

$$F_r = m\mu v$$
$$= m\mu x.$$

(iii) Periodic force.

$$= P\cos kt$$

$$\cos t = 2\pi.$$
$$\cos kT = \frac{2\pi}{K}.$$

S.H.M (1)

(i) $$\frac{d^2x_1}{dt^2} = -w_1^2 x$$

solving this eqn (i) we have,

$$x_1(t) = a_1\cos(w_1 t + \epsilon_1)$$
$$= A\cos w_1 t + B\sin w_1 t.$$
$$A = a\cos\epsilon_1, \quad B = -b\sin\epsilon_1$$

S.H.M (2)

$$\frac{d^2x_2}{dt^2} = -w_2^2 x_2$$

solving eqn (ii) we have,

$$x_2(t) = a_2\cos(w_2 t + \epsilon_2)$$

The resultant displacement from two S.H.M will be

$$x = x_1 + x_2 = a_1\cos(w_1 t + \epsilon_1) + a_2\cos(w_2 t + \epsilon_2)$$
$$= a\cos(wt + \epsilon)$$

where,
a = resultant amplitude.

ϵ_p = epoc. phase difference.

periodic
quasi periodic

11/2/2020

The Resultant of two simple Harmonic motion:

$x_1 = a_1 \cos(\omega_1 t + \epsilon_1) \rightarrow T_1 = \frac{2\pi}{\omega_1}$, where a_1 & ϵ_1 are two parameters.

$x_2 = a_2 \cos(\omega_2 t + \epsilon_2) \rightarrow T_2 = \frac{2\pi}{\omega_2}$.

can be taken as

$x = x_1 + x_2$.

$= a_1 \cos(\omega_1 t + \epsilon_1) + a_2 \cos(\omega_2 t + \epsilon_2)$.

$\boxed{= A \cos(\omega t + \epsilon)}$.

$= a_1 \cos\omega_1 t \cos\epsilon_1 - a_1 \sin\omega_1 t \sin\epsilon_1 + a_2 \cos\omega_2 t \cos\epsilon_2 - a_2 \sin\omega_2 t \sin\epsilon_2$.

$= \cos\omega_1 t\,(\underbrace{a_1\cos\epsilon_1 + a_2\cos\epsilon_2}) - \sin\omega_1 t\,(\underline{a_1\sin\epsilon_1 + a_2\sin\epsilon_2})$.

$= r\cos\theta \cos\omega_1 t - r\sin\theta \sin\omega_1 t$

$= r\cos(\omega_1 t + \theta)$.

let, $a_1\cos\epsilon_1 + a_2\cos\epsilon_2 = r\cos\theta$.

wher, r is resultanat amplitude.

$(r\cos\theta)^2 + (r\sin\theta)^2 = (a_1\cos\epsilon_1 + a_2\cos\epsilon_2)^2 + (-(a_1\sin\epsilon_1 + a_2\sin\epsilon_2))^2$

$\Rightarrow r^2 = a_1^2 + a_2^2 + 2a_1a_2\cos(\epsilon_1 \sim \epsilon_2)$

$\Rightarrow r = \sqrt{a_1^2 + a_2^2 + 2a_1a_2\cos(\epsilon_1 \sim \epsilon_2)}$

$= a_1 + a_2$ where $\epsilon_1 = \epsilon_2$.

again.

$$\frac{r\sin\theta}{r\cos\theta} = \tan\theta = \frac{a_1\sin\epsilon_1 + a_2\sin\epsilon_2}{a_1\cos\epsilon_1 + a_2\cos\epsilon_2}.$$

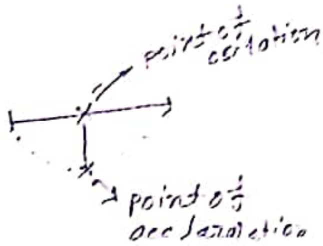
$$\Rightarrow \theta = \tan^{-1}\left[\frac{a_1\sin\epsilon_1 + a_2\sin\epsilon_2}{a_1\cos\epsilon_2 + a_2\cos\epsilon_2}\right].$$

P–51

10.(3).

amplitude a

period T.



$v^2T^2 = 4\pi^2(a^2 - x^2)$.

let, $\cancel{\frac{d^2x}{dt^2} = -w^2x}$.

x be the distance of the periodic oscillation and any time t & $v\left(\frac{dx}{dt}\right)$ is the corresponding velocity

∴ The eqn of motion can be written as

$$\frac{d^2x}{dt^2} = -w^2x$$

$$\Rightarrow v.\frac{dv}{dx} = -w^2x$$

$$\Rightarrow 2v\,dv = -w^2 2x\,dx.$$

on integration we have,

$\Rightarrow v^2 = -w^2x^2 + c$, where c is the constant of integration

Initially at $x = a$, $v = 0$

$\therefore c = w^2a^2$.

Then, $v^2 = -w^2x^2 + w^2a^2$

$= w^2(a^2 - x^2) = \frac{4\pi^2}{T^2}(a^2 - x^2)$.

13/2/2020

$\because w_1 = w_2$ $\quad w_1 - w_2 \simeq \delta$ (very small quantity).

Periodic time.

(1)

$x_1 = a_1 \cos(w_1 t + \epsilon_1)$, $x_2 = a_2 \cos(w_2 t + \epsilon_2)$.

where, $w_1 - w_2 = \delta$.

The eqn of the resultant of two SHM is $X = x_1 + x_2$

$= a_1 \cos(w_1 t + \epsilon_1) + a_2 \cos(w_1 t - \delta t + \epsilon_2)$.

δ is very small quantity we can say $\epsilon_2 - \delta t = \epsilon_3$.

$= a_1 \cos(w_1 t + \epsilon_1) + a_2 \cos(w_1 t + \epsilon_3)$.

$= A \cos(w_1 t + \epsilon_0)$

where, $A^2 = a_1^2 + a_2^2 + 2a_1 a_2 \cos(\epsilon_1 - \epsilon_3)$.

$= a_1^2 + a_2^2 + 2a_1 a_2 \cos(\epsilon_1 - \epsilon_2 + \delta t)$

$= a_1^2 + a_2^2 + 2a_1 a_2 \cos(\epsilon_1 - \epsilon_2 + (w_1 - w_2)t)$

where,

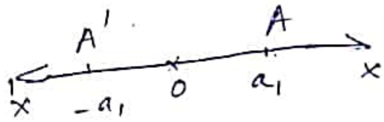
$$\tan\epsilon = \frac{a_1 \sin\epsilon_1 + a_2 \sin\epsilon_3}{a_1 \cos\epsilon_1 + a_2 \cos(\epsilon_2 - (w_1 - w_2)t)}$$

$x = A \cos w t$

at $x = 0$, $t = T/4$.

$0 = A \cos w \cdot T/4$

$\cos \pi/2 = A \cos wT/4$.



$\frac{d^2x}{dt^2} = -w^2 x$.

$\left(\frac{dx}{dt}\right)^2 = -w^2x^2 + c$.

$\frac{dx}{dt} = -w\sqrt{a^2 - x^2}$

T = time required to travel $AOA'A$.

$dt = -\frac{dx}{w\sqrt{a^2-x^2}}$

$\int_0^T dt = \frac{-2}{w}\int_a^{-a} \frac{dx}{\sqrt{a^2-x^2}}$.

$\int_0^T dt = \frac{4}{w}\int_0^a \frac{dx}{\sqrt{a^2-x^2}}$.

$= \frac{4}{w} \sin^{-1}\left(\frac{x}{a}\right)\Big]_0^a$

$= \frac{4}{w}\frac{\pi}{2} - 0$

$\Rightarrow T = \frac{2\pi}{w}$

$$x = A\cos\left(\omega t + \frac{a_1 \sin\epsilon_1 + a_2 \sin\epsilon_3}{a_1\cos\epsilon_1 + a_2\cos\epsilon_3}\right).$$

$$= A\cos\left(\cancel{\omega_1 t + \frac{a_2\delta}{a_1\cos\epsilon_1} + \epsilon_1}\right).$$

$$= A\cos(\delta t + \epsilon_1).$$

$$T = \frac{2\pi}{\delta} = \frac{2\pi}{\omega_1 - \omega_2}$$

$$\frac{(a_1\sin\epsilon_1 + a_2\sin\epsilon_3)}{a_1\cos\epsilon_1}\left(1 + \frac{a_2}{a_1}\frac{\cos\epsilon_3}{\cos\epsilon_1}\right)^{-1}$$

$$= \left[\left(\tan\epsilon_1 + \frac{a_2}{a_1}\frac{\sin\epsilon_3}{\cos\epsilon_1}\right)\left(1 - \frac{a_2\cos\epsilon_3}{a_1\cos\epsilon_1}\right)\right].$$

$$= \left[\tan\epsilon_1 - \cancel{\tan\epsilon_1}\ \frac{a_2}{a_1}\frac{\sin(\epsilon_1 - \delta t)}{\cos\epsilon_1}\right]\left[1 - \frac{a_2\cos(\epsilon_1 - \delta t)}{a\cos\epsilon_1}\right]$$

$$= \frac{a_2}{a_1\cos\epsilon_1}\ \cancel{+\infty}$$

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P-6.5 A ...

(3) As per ~~sensitive~~ the given law of force.

the eqn of SHM can be taken as

—

$$\frac{d^2x}{dt^2} = -\frac{\mu x}{a} \quad \text{--- (I)} \qquad T_1 = \frac{2\pi}{\sqrt{\mu/a}}$$

at t = 0, $x = a$, $\frac{dx}{dt} = 0$.

The motion subject to the ~~with~~ disturbing force is.

$$\frac{d^2x}{dt^2} = -\frac{\mu x}{a} - \frac{\nu x^3}{a^3}.$$

$$= \frac{-\mu a^2 x - \nu x^3}{a^3} \quad \text{--- (II)}$$

clearly the period of the undisturb oscilation or SHM.

is $T_1 = \cancel{\sqrt{\frac{2\pi}{\mu}}} \; 2\pi/\sqrt{\mu/a}$.

The period of the distr

~~From (III)~~ multiping (II) by $2\frac{dx}{dt}$ and integrating we have.

$$\frac{d^2x}{dt^2} \cdot \left(\frac{dx}{dt}\right)^2 = -\frac{\mu}{a}x^2 - \frac{\nu x^4}{2a^3} + c \quad \text{--- (III)}$$

Now, at, $x = a$ $\frac{dx}{dt} = 0$,

$$c = \frac{\mu}{a}a^2 + \frac{\nu a^4}{2a^3}$$

$$\cancel{= \left(\mu + \frac{\nu}{2}\right)a}.$$

$$\left(\frac{dx}{dt}\right)^2 = \frac{\mu}{a}(a^2 - x^2) + \frac{\nu}{2a^3}(a^4 - x^4).$$

$$= \frac{a^2 - x^2}{a}\left\{\mu + \frac{\nu}{2a^2}(a^2 + x^2)\right\}. \quad \text{--- (IV)}$$

left ~~hand si~~ most point of the osicilLation is given by.

$$\mu + \frac{\gamma}{2a^2}(a^2+x^2) = 0.$$

$$\Rightarrow \quad a^2 + x^2 = \frac{2\mu a^2}{\gamma}.$$

$$\Rightarrow \quad x^2 = -\frac{2a^2\mu}{\gamma} + a^2.$$

$$= -a^2\left(\frac{2\gamma+\gamma}{\gamma}\right).$$

$$\cancel{\frac{dx}{dt} = -\frac{a^2-x^2}{a}\frac{\gamma}{2a^2}\left(x^2 + \frac{2\mu+\gamma}{\gamma}\right)a^2}.$$

$$\frac{dx}{dt} = -\frac{a^2-x^2}{a}\frac{\gamma}{2a^2}\left(x^2 + \frac{2\mu+\gamma}{\gamma}\right).$$

18/2/2020

The periodic time for SHM (i) given by —

$$T_1 = 2\pi\sqrt{a/\mu}.$$

~~Math~~

P.

$$\frac{dx}{dt} = \pm\sqrt{(a^2-x^2)(x^2+p^2)}\sqrt{\gamma/2a^3}, \quad \text{where, } p^2 = \left(\frac{2\mu+\gamma}{\gamma}\right)a^2. \quad \text{— (v)}$$

the lower amplitude is the solⁿ of $\frac{dx}{dt} = 0$,

(apart form $x = a$).

other amplitude is

$$x^2 - a^2 = 0$$
$$x^2 = \pm a.$$

$$x^2 + p^2 = 0$$
$x = \pm ip$ is not real or feasible.

Let, T_2 be the periodic time for the disturb oscillation.

Then,

$$T_2 = 2\int_0^{T_2} dt$$

$$= -2\int_{-a}^{a} \sqrt{2a^3/\gamma}\ \frac{dx}{\sqrt{(a^2-x^2)(x^2+p^2)}}$$

$$= -4\sqrt{\frac{2a^3}{\gamma}}\int_0^a \frac{dx}{\sqrt{(a^2-x^2)(x^2+p^2)}}$$

Let, $a^2-x^2 = y^2$.

x	0	a
y	a	0

$-2x\,dx = 2y\,dy$.

$a^2-x^2=y^2$.

$\Rightarrow -x\,dx = y\,dy$.

$\cancel{x = a^2 - y^2}$ $\quad x = \sqrt{a^2-y^2}$

$$= 4\sqrt{2a^3/\gamma}\int_a^0 \frac{y\,dy}{y\sqrt{(a^2-y^2+p^2)}}$$

$$T_2 = -4\sqrt{\frac{2a^3}{\gamma}}\left\{1-\frac{1}{4}\left(\frac{a}{p}\right)^2\right\}\frac{\pi}{2p}.$$

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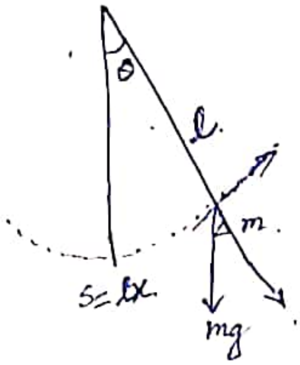
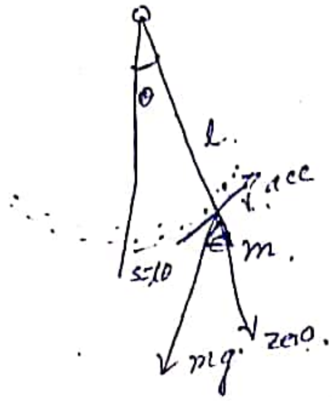
6 (a)

Tangential acceleration . (REF for unit mass).

$$= \frac{d^2 s}{dt^2} = \frac{d^2}{dt^2}(l\theta)$$

$$= l\frac{d^2\theta}{dt^2}.$$





RE.F ,

$$m\frac{d^2 s}{dt^2}$$

$$= ml\frac{d^2\theta}{dt^2}.$$

the resdve part of the gravitational force along the tangential . . . is . . $mg\cos(\pi/2+\theta) = -mg\sin\theta$.

Third force is the ~~velo resting~~ resisting force, which proposonal to the velocity $\frac{ds}{dt} = l\frac{d\theta}{dt}$.

$$\therefore F_R = Rl\frac{d\theta}{dt}.$$

Considering the all the forcess, the eqn of motion of the pendulam is —

$$ml\frac{d^2\theta}{dt^2} = -mg\sin\theta - F_R.$$

$$\frac{d^2\theta}{dt^2} = -g/l\sin\theta - \frac{Rl}{ml}\frac{d\theta}{dt}.$$

$$= -g/l\sin\theta - R/m\frac{d\theta}{dt}.$$

or, $\frac{d^2\theta}{dt^2} + \frac{R}{m}\frac{d\theta}{dt} + g/l\sin\theta = 0$, as θ is very very small quantity and measured in radial scale.

$\Rightarrow \frac{d^2\theta}{dt} + 2K\frac{d\theta}{dt} + g/l\,\theta = 0$, $2K = C/m$, $\sin\theta = \theta$.

(4)

Let, $\theta = e^{\lambda t}$ be the soln of the S.H.M. given by -

$$\frac{d^2\theta}{dt^2} + 2k\frac{d\theta}{dt} + g/l\,\theta = 0.$$

Then, the auxiliary eqn.

$$\lambda^2 + 2k\lambda + \frac{g}{l} = 0.$$

$$\Rightarrow \lambda = \frac{-2k \pm \sqrt{4k^2 - 4g/l}}{2}$$

$$= -k \pm \sqrt{k^2 - g/l}$$

$$= -k \pm i\sqrt{g/l - k^2}, \text{ where, } k^2 < g/l.$$

Tautochronous $\rightarrow$ $w(t)$.
isochronous $\rightarrow$ $\cancel{w(t)}$
w is not fn of t.

Then,

$$\theta(t) = Ae^{-kt}\cos\left(\left(\sqrt{g/l - k^2}\right)t + B\right), \text{ where}$$

A and B are arbitrary constant.

$$T = \frac{2\pi}{\sqrt{g/l - k^2}}.$$

(5)(a) Eco Fe FR

$$m\frac{d^2x}{dt^2} = -mn^2x^2 - mn\frac{dx}{dt}, \; x = \text{distance}.$$

PF

$$+ mg\cos pt.$$

Free vibration, $m\frac{d^2x}{dt^2} = -mn^2x - mn\frac{dx}{dt}$. —(i)

$$m\frac{d^2x}{dt^2} = -mn^2x - mn\frac{dx}{dt} + mg\cos pt. \quad \text{—(ii)}$$

from (11) $\frac{d^2x}{dt^2} + n\frac{dx}{dt} + n^2x = 0$.

let, $e^{+\lambda t}$ be the solⁿ, then.

$$\lambda^2 + n\lambda + n^2 = 0.$$

$$\lambda = \frac{-n \pm \sqrt{n^2 - 4n^2}}{2}.$$

$$= \frac{-n \pm i\sqrt{3n^2}}{2}.$$

$$\therefore x = C_1 e^{-n/2\, t} \cos\left(\frac{\sqrt{3n^2}}{2}t + c_2\right).$$

$$T_1 = \frac{2\pi}{n\sqrt{3}/2} = \frac{4\pi}{\sqrt{3}n}.$$

$$\frac{d^2x}{dt^2} + n\frac{dx}{dx} + n^2x = g\cos pt.$$

$$T_2 = \frac{2\pi}{p}.$$

$$\frac{4\pi}{\sqrt{3}\cdot n} = \frac{1}{2}\frac{2\pi}{p}.$$

$$\Rightarrow 3n^2 = 16p^2.$$

$$x(t) = C_1 e^{-n/2\, t}\cos\left(\frac{\sqrt{3}n}{2}t + c_2\right)$$

$$+ PI.$$

$$PI = \frac{g\cos pt}{D^2 + nD + n^2},$$

$$= \frac{-g\cos pt}{(n^2 - p^2) + nD}.$$

$$= -\frac{g\{(n^2-p^2) - nD\}}{(n^2-p^2)^2 - n^2D^2}\cos pt.$$

$$= -\frac{g(n^2 - p^2 - nD)}{(n^2-p^2)^2 - n^2p^2}\cos pt.$$

$$= -\frac{g\left[(n^2-p^2)\cos pt + np\sin pt\right]}{(n^2-p^2)^2 + n^2p^2}$$

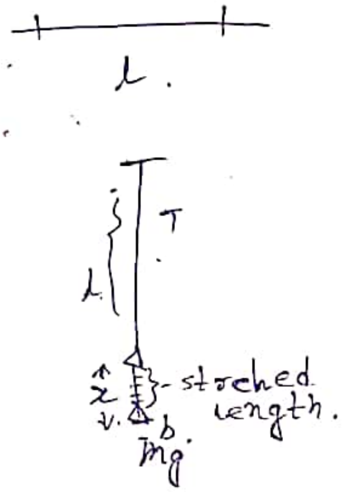
$$= -g\frac{A\cos(pt+\epsilon)}{n^4 - n^2p^2 + p^4}.$$

g.

20/12/20

(8)

Eqn of motion.



$$m\frac{d^2x}{dt^2} = mg - T$$

T is the tention of string caused by the weight..

$$T = \lambda\frac{x}{l}$$

λ coefficient of elasticity..

$$m\frac{d^2x}{dt^2} = mg - T$$

$$= mg - \lambda\frac{x}{l}$$

$$= mg - \frac{mgl}{b}\frac{x}{l}$$

$$= mg - \frac{mgx}{b}$$

$$= mg\left(1 - \frac{x}{b}\right).$$

$$\Rightarrow \frac{d^2x}{dt^2} = g\left(1 - \frac{x}{b}\right).$$

T = weight

$$T = \frac{\lambda b}{l} = mg.$$

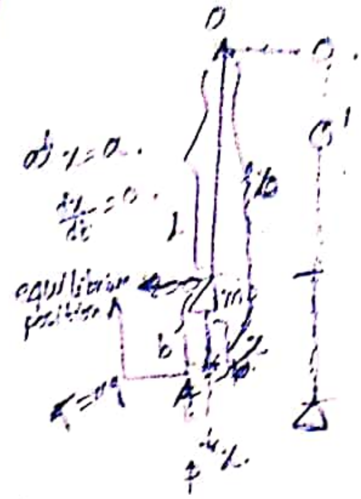
$$\Rightarrow \lambda = \frac{mg}{b}l.$$

1.a)

The free eqn of motion is given by

$$m\frac{d^2x}{dt^2} = mg - T$$

$$= mg - \frac{\lambda(x-l)}{l}.$$



let,

O be the fixed the point at which the string is weight less l is suspended and the bottom a bid of weight m is.

let, p be the position, free end pt of the string at any time t.

where, $OP = l + b + x$.

$$\therefore \text{(tension)}\ T = \lambda\frac{(l+b+x-l)}{l}.$$

$$= \lambda\frac{(b+x)}{l}, \text{ where, } \lambda \text{ be}$$

$T = mg$, at the position A (ie. $x=b$).

$$\Rightarrow T = \lambda\frac{b}{l} = mg.$$

$$\Rightarrow \lambda = \frac{mgl}{b}.$$

The free forced eqn of motion.

$$m\frac{d^2x}{dt^2} = mg - \frac{\lambda(b+x)}{l}$$

$$= mg - \frac{mgl}{b}\cdot\frac{(b+x)}{l}$$

$$\Rightarrow \cancel{= mg - mgl}$$

$$= \frac{mglx}{bl}. = \frac{mgx}{b}.$$

$$\frac{d^2x}{dt^2} = -\frac{gx}{b}.$$

$$= -w^2 x, \quad w^2 = \frac{g}{b}. \text{ —— (1)}$$

$$T_1 = \frac{2\pi}{w} = 2\pi\sqrt{b/g}.$$

A.

The eqn of motion with the oscillation.

$$m\frac{d^2}{dt^2}(b+x+a\sin pt) = mg - T'$$

$$= mg - \lambda\frac{(b+x+\cancel{a\sin pt})}{l}.$$

$$\Rightarrow m\frac{d^2}{dt^2}(b+x+a\sin pt) = mg - \frac{\lambda(b+x+\cancel{a\sin pt})}{l}.$$

$$\Rightarrow m\left(\frac{d^2x}{dt^2} + a\sin pt.p^2\right) = mg - \frac{\lambda(b+x+\cancel{a\sin pt})}{l}.$$

$$\Rightarrow m\left(\frac{d^2x}{dt^2} - a\sin pt.p^2\right). = \left(mg - \frac{\lambda b}{l}\right) - \lambda\left(\frac{x\cancel{+a\sin pt}}{l}\right)$$

$$= -\frac{\lambda x}{l}.$$

$$\Rightarrow m\frac{d^2x}{dt^2} = -\frac{g}{b}mx + ap^2\sin pt.$$

$$\frac{d^2x}{dt^2} = -g/b\, mg + ap^2\sin pt.$$

The solⁿ of the force.

$$x(t) = A\cos(wt+B) + \frac{ap^2}{(p^2+w^2)}\sin pt.$$

the ~~addl~~ contribution of the particle P

$\therefore$ ~~extra~~ displacement due to forced oscillation is,

$$= \frac{a\, p^2 \sin pt}{p^2 + w^2}$$

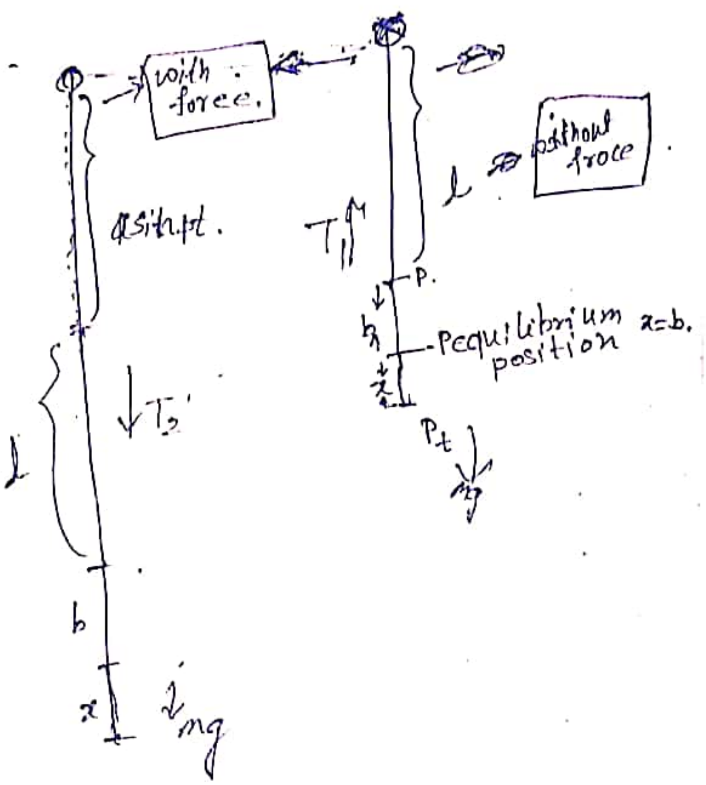
$$d. = \frac{a\, p^2 \sin pt}{w^2 - p^2} \qquad T_1 = \frac{2\pi}{w}. \qquad T_2 = \frac{2\pi}{p}.$$

$$= \frac{a \frac{4\pi^2}{T_2^2}}{\frac{4\pi^2}{T_1^2} - \frac{4\pi^2}{T_2^2}} \sin pt \;\; \frac{2\pi}{T_2} t.$$

$$= \frac{a\, 4\pi^2/T_2^2}{\frac{T_2^2 4\pi^2 - 4\pi^2 T_1^2}{T_1^2 T_2^2}}$$

$$= \frac{\frac{T_1^2 T_2^2}{T_2}}{T_2^2 - T_1^2}$$

$$= \frac{T_1}{T_2^2 - T_1^2}.$$



let, x be the current displacement of the particle.

Work - Power - Energy

Work $\xrightarrow{}$ scalar, $= W$ = workdone by the (external) applied force causing some displacement.

$\xrightarrow{}$ Fixed forced workdone.

(i) $W = \vec{F} \cdot \vec{d}$.

$$W(x) = \int_{x=0}^{x} F(x) \, dx.$$

(ii) Power $= P$ = rate of doing work.

$$= \frac{W}{T}$$

$$= \frac{F \cdot S}{T}, \quad F = \text{fixed forced}$$

$$= F \cdot \left(\frac{S}{T}\right), \quad S = \text{displacement}.$$

$$= F \cdot v.$$

(iii) Energy = = (ability) capacity of doing work.

= Mechanical ~~eigre~~ energy + potential Energy.

$= K.E + P.E.$

$= \frac{1}{2} M v^2 + P.E.$

$\boxed{\text{P. Energy is position dependent.}}$

S.H.M

l.

$$T = \frac{\lambda \cdot x}{l} = F(x).$$

work done due to the tension T from $x = x_1$ to x_2 is given by -

$$W = \int_{x_1}^{x_2} F(x) \cdot dx$$

$$= \int_{x_1}^{x_2} \frac{\lambda x}{l}\, dx. \qquad \lambda = \text{coefficient of tension.}$$

$$= \frac{\lambda}{2l}\left(x_2^2 - x_1^2\right).$$

$$= \frac{\lambda}{2l}\left(x_2 + x_1\right)\left(x_2 - x_1\right).$$

$$= \frac{1}{2}\left(\frac{\lambda x_2}{l} + \frac{\lambda x_1}{l}\right)\left(x_2 - x_1\right)$$

$$= \frac{1}{2}\left(T_2 + T_1\right) d.$$

d = displacement;

$$= \left(\frac{T_1 + T_2}{2}\right) \cdot d. \qquad T_2 = \frac{\lambda x_2}{l}, \quad T_1 = \frac{\lambda x_1}{l}.$$

Horse power:

$mg = F = \text{wt}.$

$H = 550g \times 1ft$ $\quad g = \text{ft.p.s.}$

1ft.

550 Lb.

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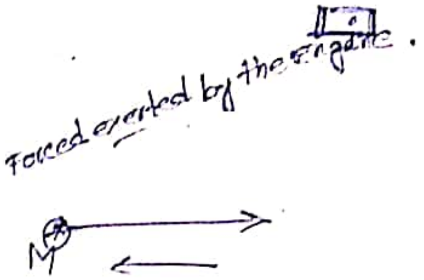
22
13, 14, 15, 16, 17

13
(a) $F = R$

$t = ?$ when $v = V$.



p = Power = R.

Friction F_{fr} =

max $v(t) = w$

$t \in I$.

$v = w$ when

$\frac{dv}{dt} = 0$,

let,
F be the force exerted by the engine, and v the velocity of the car at any time t, current distance is x

$\therefore$ the eqn of motion of the car.

$$M \cdot v \frac{dv}{dx} = F - F_{fr} \quad , \text{where}$$

$$R = F \cdot v$$

$$= \frac{R}{v} - F_{fr} \quad \text{—(1)} \qquad F = \frac{R}{v} ..$$

according to question at $v = w$, $\frac{dv}{dt} = \frac{dv}{dx} = 0$,

$$\therefore M \frac{dv}{dt} = \frac{R}{v} - F_{fr}$$

$$\Rightarrow 0 = \frac{R}{w} - F_{fr}$$

$$\Rightarrow F_{fr} = R/w.$$

$\therefore$ Eqn —(1) extendent to the following —

$$M.v\frac{dv}{dx} = R\left(\frac{1}{v} - \frac{1}{w}\right).$$

$$\Rightarrow dx = \frac{MW}{R}\frac{v^2}{(W-v)}\,dv. \quad \text{(ii)}$$

let, d be the distance of travelled by the car, when it acheves the velocity V

Integrating (ii) within the proper limits

at $x=0$, $v=0$

at $x=d$, $v=V$.

$$\int_0^d dx = \int_0^V \frac{MW}{R}\frac{v^2}{(W-v)}dv$$

$$\Rightarrow d = \frac{MW}{R}\int_0^V \frac{v^2}{(W-v)}dv.$$

$$= \frac{MW}{R}\int_0^V \frac{-(v^2+W^2)+W^2}{W-v}dv.$$

$$= \frac{MW}{R}\left[\int_0^V -(W+v)\,dv + W^2\int_0^V \frac{1}{W-v}dv\right]$$

$$= \frac{MW}{R}\left\{-\left[Wv+\frac{v^2}{2}\right]_0^V + W^2\ln(W-v)\right]_0^V$$

$$= -\frac{MW^2V}{R} - \frac{MWV^2}{R2} + \frac{MW^3}{R}\ln\left(\frac{W}{W-V}\right).$$

$$= \frac{MW^3}{R}\left[\ln\left(\frac{W}{W-V}\right) - \frac{V}{W} - \frac{1}{2}\frac{V^2}{W^2}\right]$$

1 horse-power of the engine, $H = \frac{R.V}{550}$, R.V is foot-pounds.

P-82

3/8/2020

(9) capacity of the steamer 550gH.

Mass of the steamer, M ton.

= 2240M lbs.

1 qr = 28 lbs

1 cwt = 4 qr

1 ton = 20 cwt.

= 20×4 qr

= 20×4×28

= 2240 lbs.

$$\begin{array}{r} 28 \\ \times 80 \\ \hline 2240 \end{array}$$

1000 kg = 2240 lbs = 1 ton.

0.436 kg = 1 lbs.

let, v be the current velocity of the steamer at any time t.

then, the eqn of motion can be written as

$$2240M \frac{dv}{dt} = F - R \quad \text{—(I)}$$

where, F is the force exerted by the engine.

R = the resistance offered by the surface of the water,

According to question,

$$R \propto v^2.$$

$$R = kv^2 \quad \text{—(II)}$$

k is variation constant.

from the defn of the horse power (HP), we have,

$$F.v_{max} = 550gH. \quad \text{—(III)}$$

$$\Rightarrow F.V = 550gH \Rightarrow F = \frac{550}{V}gH \quad \text{—(III)}$$

Thus eqn (I) becomes.

$$2240M \frac{dv}{dt} = \frac{550}{v}gH - kv^2 \quad \text{—(IV)}$$

Since V is the maximum of v $\forall t \in R_f$.

$$\therefore \quad \frac{550gH}{V} - KV^2 = 0$$

, $v_{max} = V$ is constant.
then, $\frac{dv}{dt} = 0$.

$$\Rightarrow \quad K = \frac{550}{V^3} gH$$

Hence, the simplified the eqn of motion becomes:

$$224 0M \frac{dv}{dt} = \frac{550gH}{V} - \frac{550}{V^3} gH \cdot v^2.$$

$$= \frac{550gH}{V^3} (V^2 - v^2)$$

$$\Rightarrow \frac{dv}{dt} = \frac{55gH}{224} \frac{1}{MV^3} (V^2 - v^2)$$

at $t = 0$, $v = 0$

and $t = t$, $v = v(t)$.

$$\Rightarrow dt = \frac{224 \quad MV^3}{55gH} \frac{dv}{(V^2 - v^2)}$$

Integration, between the proper limit.

$$\int_0^t dt = \int_0^v \frac{224}{55gH} MV^3 \frac{dv}{(V^2 - v^2)}.$$

$$\Rightarrow t = \frac{224MV^3}{55gH} \int_0^v \frac{1}{(V+v)(V-v)} dv$$

$$\Rightarrow t = \frac{224MV^3}{55gH} \frac{1}{2V} \left[\log \frac{V+v}{V-v} \right]_0^v$$

$$t = \frac{112}{55} M V^2 \ln \left| \frac{V+v}{V-v} \right|$$

(15)

M = mass of the ~~b~~ train., let v be the velocity of the train at any time t. then,

H = ~~capacity~~. $P \cdot v$. where P be the force generated by the engine of capacity H.

$$\Rightarrow \quad P = \frac{H}{v}.$$

let,

F = Resistance (constant).

Thus, we have

$$M \frac{dv}{dt} = P - F$$

$$= \frac{H}{v} - F.$$

$$\Rightarrow \frac{dv}{dt} = \frac{H}{Mv} - \frac{F}{M}.$$

Let, T be the required the time to attain the velocity V.

$$\Rightarrow \int_0^V \frac{Mv}{H - Fv} \cdot dv = \int_0^T dt$$

$$\Rightarrow [t]_0^t = -\frac{M}{F} \int_0^V \left(\frac{-Fv + H - H}{H - Fv} \right) dv$$

$$= -\frac{M}{F} \left[\int_0^V dv - \int_0^V \frac{H}{H - Fv} dv \right]$$

$$= -\frac{M}{F} \cdot V + \frac{MH}{F^2} \left(-\ln(H - FV) + \ln H \right)$$

$$= -\frac{MV}{F} + \frac{MH}{F^2} \ln \left(\frac{H}{H - FV} \right)$$

Let m be the mass of the [illegible] string.

$\therefore$ mass of the string per unit length $= \frac{m}{2l}$ ($(l+a+l-a)$) ($2l$ is total length of the string)

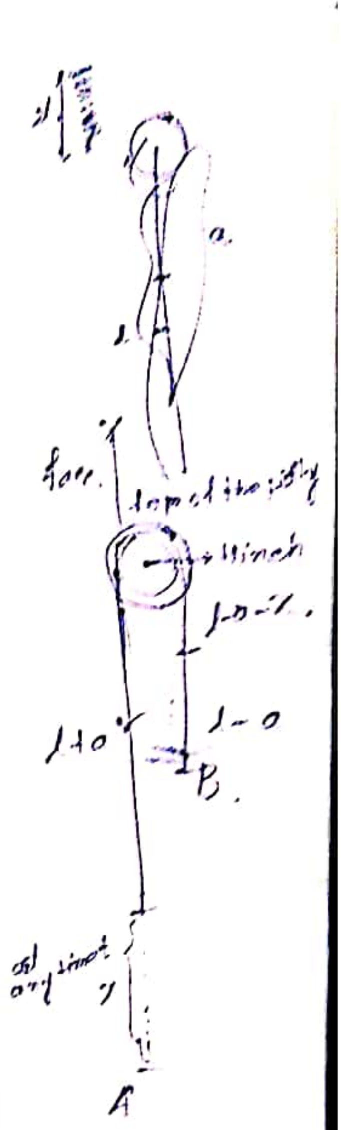
smooth

Let T be the tension acting toward the fixed

$\therefore$ the eqn of motion of the longer end (A) is

$$M_L\left(\frac{d^2}{dt^2}(l+x+a)-f\right) = M_L g - T \quad \text{—(i)}$$

$$M_L = \frac{m(l+a+x)}{2l}.$$



Similarly

The eqn of motion of the shorter end is

$$\cancel{M_s \frac{d^2}{dt^2}(\quad x + f)}$$
$$M_s\left(\frac{d^2}{dt^2}(l-a-x) - f\right) = M_s g - T.$$

$$M_s\left(\frac{d^2}{dt}x + f\right) = T - M_s g \quad \text{—(ii)}$$

where $M_s = \frac{m(l-a-x)}{2l}$.

$$\cancel{\Rightarrow M_s \frac{d^2x}{dt} +}$$

Eleminating T from (i) & (ii) we have,

$$\frac{m(l+a+x)}{2l}(\ddot{x}-f) + \frac{m(l-a-x)}{2l}(\ddot{x}+f) = \frac{m(l+a+x)}{2l}g - \frac{m(l-a-x)}{2l}\cdot g$$

$$= \frac{m(a+x)}{l}g$$

$$\Rightarrow \left[\quad l \quad\right]\ddot{x} - (a+x)f = (a+x)g$$

:

$$l \frac{d^2x}{dt^2} \cancel{+ a \mu x} = (a+x)(f+g)$$

$$\Rightarrow \frac{d^2x}{dt^2} = \frac{f+g}{l}(a+x)$$

Multipling both side by $2\frac{dx}{dt}$ & integrating, we have.

$$\left(\frac{dx}{dt}\right)^2 = \mu (a+x)^2 + c.$$

initially $t=0,\ x=0$.

$$\therefore c = -\mu a^2$$

$$\frac{dx}{dt} = \sqrt{\mu(a+x)^2 - \mu a^2}$$

$$= \sqrt{\mu}\sqrt{(a+x)^2 - a^2}$$

$$= \cancel{\sqrt{\mu}}$$

$$\int_0^T dt = \sqrt{\frac{l}{f+g}} \int_0^{l-a} \frac{d(a+x)}{\sqrt{(a+x)^2 - a^2}}$$

$$\begin{cases} l-a-x=0 \\ x = l-a \end{cases}$$

$$\Rightarrow T = \sqrt{\frac{l}{f+g}} \ln\left|(a+x) + \sqrt{(a+x)^2 - a^2}\right|_0^{l-a}$$

$$\Rightarrow T = \sqrt{l/f+g}\ \ln\left|l + \sqrt{l^2 - a^2}\right| - \ln a.$$

$$= \sqrt{l/f+g}\ \ln\left|\frac{l+\sqrt{l^2-a^2}}{a}\right|.$$

are uniform.

(i) $F = m.f = m\left(\frac{v-u}{t}\right)$.

(ii) $F \times t = (mv - mu) \rightarrow$ change of momentum.

(Impulse of the force). $\quad$ final momentum $\quad$ initial momentum.

(iii) changes of kinetic energy = by work done by the external force/forces applied to the body.

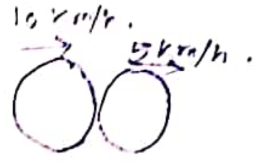
$F = mf$. , f is variable.

$\Rightarrow F(x) = mv\frac{dv}{dx}$.

$\Rightarrow \int_{u_{initial}}^{v_{final}} mv\, dv = \int_{x_{in}}^{x_{final}} F(x)\, dx$. = work done.

$\Rightarrow \frac{1}{2}mv^2 - \frac{1}{2}mu^2 = W.D.$

Collision of Elastic/inelastic Bodies :-



(i) Conservation of linear momentum :-

Let, m & M be two elastic bodies $(0 < e \leq 1)$ colided directly and with velocitys u_1 and u_2. and seperated with the velocityes v_1 and v_2 respectively.

Then, The total momentum before impacted = total momentum after impact.

$\rightarrow m u_1 + M u_2 = m v_1 + M v_2$.

(ii) Conservation of kinetic energy :-

$$\frac{1}{2} m u_1^2 + \frac{1}{2} M u_2^2 = \frac{1}{2} m v_1^2 + \frac{1}{2} M v_2^2 .$$

(iii) Newton experimental law (for Elastic bodies):

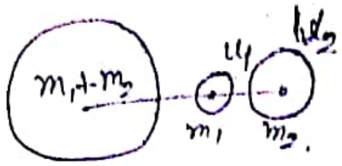
velocity of separation after impact

= e × (velocity of approch).

$$\Rightarrow v_2 - v_1 = e \times (u_1 - u_2).$$

(19) P-[illegible]

Let, V be the velocity of the mass (m_1+m_2) before explosion. Then by the conservation of momentum we have.



internal explosion.

$$(m_1+m_2)V = m_1u_1 + m_2u_2$$

$u_1 - u_2 = ?$

Again, E be the explotion.
form o conservation of kinetic energy of we have

$$E + \frac{1}{2}(m_1+m_2)V^2 = \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2.$$

$$\Rightarrow E = \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 - \frac{1}{2}(m_1+m_2)\left(\frac{m_1u_1+m_2u_2}{m_1+m_2}\right)^2.$$

$$\Rightarrow E = \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 - \frac{1}{2}\frac{(m_1u_1+m_2u_2)^2}{m_1+m_2}.$$

$$\Rightarrow E = \frac{1}{2(m_1+m_2)}\left((m_1+m_2)(m_1u_1^2 + m_2u_2^2) - (m_1u_1+m_2u_2)^2\right)$$

$$= \frac{1}{2(m_1+m_2)}\left[\cancel{m_1^2u_1^2} + m_2^2u_2^2 - \cancel{m_1^2u_1^2} - 2m_1m_2u_1u_2 - m_2^2u_2^2 + m_1m_2u_2^2 + m_1m_2u_1^2\right]$$

$$= \frac{1}{2(m_1+m_2)}\left[m_1m_2\left[u_1^2 + u_2^2 - 2u_1u_2\right]\right]$$

$$= \frac{m_1m_2}{2(m_1+m_2)}R_{rel}^2$$

R = Relative velocity.

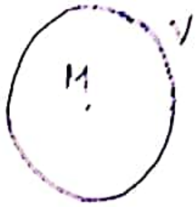
$(u_1-u_2)^2$.

Relative Velocity :-

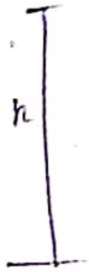
$$R_{vel} = \pm\sqrt{\frac{2E(m_1+m_2)}{m_1 m_2}}$$

(16)

recoil : opposite direction



(17)



$E = mgh$

let,
M be the mass of the gun
and m be the mass of the shell
let, u be the velocity of the shell and v be the recoil of the gun.
by the conservation of momentum, we have

$$(M+m)\cdot 0 = Mv + mu.$$

$$\Rightarrow u = -\frac{Mv}{m}.$$

$$E + 0 = \frac{1}{2}Mv^2 + \frac{1}{2}mu^2.$$

$$\Rightarrow mgh = \frac{1}{2}Mv^2 + \frac{1}{2}mu^2$$

$$= \frac{1}{2}Mv^2 + \frac{1}{2}m\cdot\frac{M^2v^2}{m^2}$$

$$= \frac{1}{2}Mv^2 + \frac{1}{2}\frac{M^2v^2}{m}$$

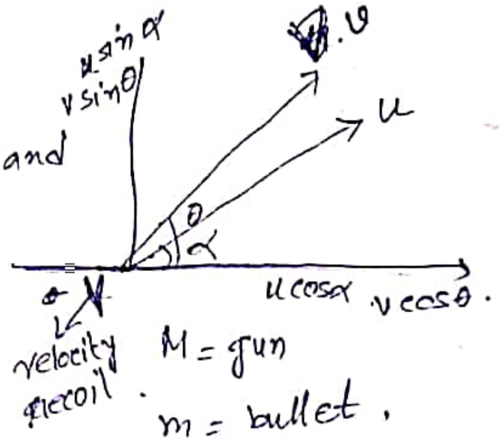
$$\Rightarrow mgh = \frac{1}{2}Mv^2\left(1+\frac{M}{m}\right).$$

$$\Rightarrow 2mgh = Mv^2\left(\frac{m+M}{m}\right).$$

$$\Rightarrow v^2 = \frac{2m^2gh}{M(m+M)}.$$

$$\Rightarrow v = \sqrt{\frac{2m^2gh}{M(m+M)}}.$$

20) Let, M be the mass of the gun and m be the mass of the bullet.



M = gun

m = bullet.

$$\frac{M}{m} = n.$$

The resolve parts of the velocity component,

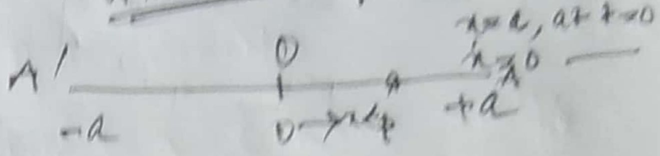
From the conservation of linear momentum

$(m+M)\cdot = MV + m\cdot$

Problem.
(13) A particle is moving in a straight line under S.H.M of amplitude a & period T, when in a position of rest is given a blow which imparts a velocity u towards the mean center. Show that it will arrive at its next position of instantaneous rest at time less by $\frac{T}{\pi}\tan^{-1}\left(\frac{uT}{2\pi a}\right)$ than if it had not received the impulse. Show that it will continue simple Harmonic Motion of same period but of amplitude $\left(a^2+\frac{u^2T^2}{4\pi^2}\right)^{1/2}$.

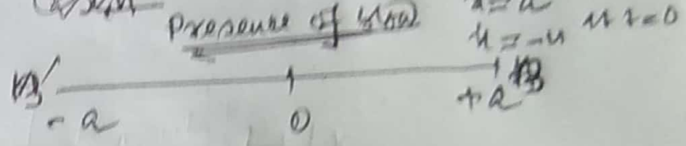
Case I

Without impulsive blow



Case II

Presence of blow



Figures;

$T = 2\pi/\omega$

Soln

The general eqn of S.H.M without any form of damping or periodic force is given by

$$\frac{d^2x}{dt^2} = -\omega^2 x \quad \text{——— (I)}$$

the soln of (I) can be written as

$$x(t) = A\cos\omega t + B\sin\omega t \quad \text{——— (II)}$$ where A & B are arbitrary constants.

For the case I (described in the Figure), we have

at $t=0$, $x=a$ & $\dot{x}=0$, which gives

$$A = a \quad \& \quad B = 0$$

$$\therefore x(t) = a\cos\omega t \quad \text{——— (III)}$$

Let the particle will come back to the next instantaneous position A' $(x=-a)$ at time t_1 (say).

then $-a = a\cos\omega t_1$

or, $\cos\omega t_1 = -1 = \cos\pi$, where period $T = 2\pi/\omega$.

$\Rightarrow t_1 = \pi/\omega = T/2$.

For the 2nd case (subject to impulse imparting velocity u towards fixed pt.),

we have from eqn - (II)

$x(t) = A\cos\omega t + B\sin\omega t,$

together with $x = a$ & $\dot{x} = -u$ at $t=0$.

thus, $A = a$, &

$$-u = -A\omega\sin\omega 0 + B\omega\cos\omega.0$$

$$= 0 + B\omega$$

$$\Rightarrow B = -u/\omega$$

$$\therefore x(t) = a\cos\omega t - u/\omega \sin\omega t \quad \text{— (IV)}$$

Let it will come back to the 2nd resting (instantaneous) place B' (figure) at time t_2 (say). then we have from (IV)

$$-a = a\cos\omega t_2 - u/\omega \sin\omega t_2$$

or, $a(1+\cos\omega t_2) = u/\omega \sin\omega t_2$

$$\Rightarrow a \cdot 2\cos^2\left(\frac{\omega t_2}{2}\right) = u/\omega \cdot 2\sin\left(\frac{\omega t_2}{2}\right)\cancel{\cos\left(\frac{\omega t_2}{2}\right)}$$

or, $\tan\left(\frac{\omega t_2}{2}\right) = \frac{a\omega}{u}$

$$\Rightarrow \cancel{t_2}$$

$$\Rightarrow t_2 = 2/\omega \tan^{-1}\left(\frac{a\omega}{u}\right) = \frac{T}{\pi}\tan^{-1}\left(\frac{2\pi a}{uT}\right)$$

Hence the lesser time is

$$T^* = t_1 - t_2 = T/2 - T/\pi \tan^{-1}(2\pi a/uT)$$

$$= T/\pi\left[\pi/2 - \tan^{-1}\left(\frac{2\pi a}{uT}\right)\right]$$

$$= T/\pi \cot^{-1}\left(\frac{2\pi a}{uT}\right)$$

$$= T/\pi \tan^{-1}\left(\frac{uT}{2\pi a}\right).$$

Since $\tan^{-1}\theta + \cot^{-1}\theta = \pi/2$ & $\cot^{-1}\theta = \tan^{-1}(1/\theta)$

Equation (v) can be written as

$$x(t) = R\cos\theta\cos\omega t + R\sin\theta\sin\omega t$$

where $R\cos\theta = a$ & $R\sin\theta = +u/\omega$

$$= R\cos(\omega t + \theta),$$

Amplitude of 2nd motion is

$$\max x(t) = R = \sqrt{a^2 + u^2/\omega^2}$$

$$= \sqrt{a^2 + \frac{u^2T^2}{4\pi^2}}.$$

(1) A particle is executing S.H.M of amplitude a under an attraction $\mu x/a$. If a small disturbing force $\lambda x^3/a^3$ towards the center be introduced (the amplitudes being unaltered), show that the period, to the first approximation, is decreased in the ratio $\left(1 - \frac{3\lambda}{8\mu}\right):1$.

Ans:

The eqn of the S.H.M under the force of attraction $\mu x/a$ (controlling force) is

$$\frac{d^2x}{dt^2} = -\frac{\mu x}{a}.$$

Therefore $T_1 = \frac{2\pi}{\sqrt{\mu/a}}$.

The eqn of motion under the attraction & the small disturbing force is given by

$$\frac{d^2x}{dt^2} = -\frac{\mu x}{a} - \frac{\lambda x^3}{a^3}.$$

Multiplying both sides by $2\frac{dx}{dt}$ & integrating we have

$$\left(\frac{dx}{dt}\right)^2 = -\frac{\mu}{a}x^2 - \frac{\lambda}{2a^3}x^4 + c_1.$$

By using the initial conditions

at $x = a, \frac{dx}{dt} = 0$, we have

$$c_1 = \frac{\mu}{a}\cdot a^2 + \frac{\lambda a^4}{2a^3} = \mu a - \frac{\lambda}{2}a$$

$$\therefore \left(\frac{dx}{dt}\right)^2 = \frac{\mu}{a}(a^2 - x^2) + \frac{\lambda}{2a^3}(a^4 - x^4)$$

$$= (a^2 - x^2)\left\{\frac{\mu}{a} + \frac{\lambda}{2a^3}(a^2 + x^2)\right\}$$

$$= \frac{\mu}{a}(a^2 - x^2)\left\{1 + \frac{\lambda}{2\mu a^2}(a^2 + x^2)\right\}$$

$$\therefore \frac{dx}{dt} = -\sqrt{\mu/a}\,\sqrt{a^2-x^2}\left[1+\frac{\nu}{2\mu a^2}(a^2+x^2)\right]^{1/2}$$

$$\therefore dt = -\sqrt{a/\mu}\,\frac{dx}{\sqrt{a^2-x^2}\left\{1+\frac{\nu}{2\mu a^2}(a^2+x^2)\right\}^{1/2}}$$

$$= -\sqrt{a/\mu}\,\frac{1}{\sqrt{a^2-x^2}}\left[1+\frac{\nu}{4a^2\mu}(-a^2-x^2)\right]dx \quad **$$

Note: $(1+X)^{-1/2} \approx \left[1-\frac{1}{2}X\right]$ neglecting higher order term. Here $X = \frac{\nu}{2\mu a^2}(a^2+x^2)$.

Integrating between the proper limits (at $t=0$, $x=a$ & at $t=t_1$, $x=0$, we have

$$\int_0^{t_1} dt = \sqrt{a/\mu}\int_0^a\left[\frac{1}{\sqrt{a^2-x^2}}\left(1-\frac{\nu a^2}{2\mu a^2}\right)+\frac{\nu}{4\mu a^2}\sqrt{a^2-x^2}\right]dx$$

$$\Rightarrow t_1 = \sqrt{a/\mu}\left[\left(1-\frac{\nu}{2\mu}\right)\cdot\sin^{-1}x/a\Big|_0^a+\frac{\nu}{4\mu a^2}\left\{\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}x/a\right\}\Big|_0^a\right]$$

$$= \sqrt{a/\mu}\left[\left(1-\frac{\nu}{2\mu}\right)\cdot\frac{\pi}{2}+\frac{\nu}{4\mu a^2}\cdot\frac{a^2}{2}\cdot\frac{\pi}{2}\right]$$

$$= \sqrt{a/\mu}\left[1-\frac{\nu}{2\mu}+\frac{\nu}{8\mu}\right]\frac{\pi}{2}$$

$$= \sqrt{a/\mu}\left(1-\frac{3\nu}{8\mu}\right)\frac{\pi}{2}$$

$$\therefore T_2 = 4t_1 = 2\pi\sqrt{a/\mu}\left(1-\frac{3\nu}{8\mu}\right)$$

$$\& \quad T_1 = 2\pi\sqrt{a/\mu}.$$

$$\therefore T_2 : T_1 = \left(1-\frac{3\nu}{8\mu}\right) : 1.$$

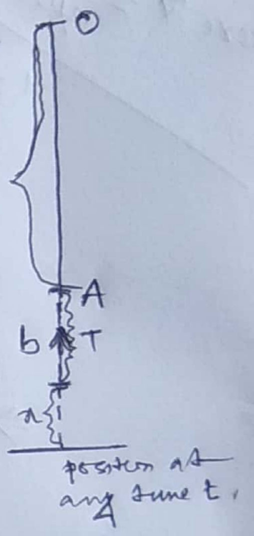
2. A particle is suspended at the end of an elastic string of negligible mass. At time $t=0$, when the particle is in equilibrium, the point of suspension begins to move so that its downwards displacement at any time t is $a\sin pt$, (a is a constant). Show that the contribution of forced oscillation to the vertical displacement is $\frac{T_2^2}{T_2^2-T_1^2} a\sin pt$, where T_1 & T_2 are the periods of free & forced oscillations.

Sol.:

Let the unstretched length of the string be 'l' & 'b' is the extension of the equilibrium point. Then the mass of the string will be ballanced by the generated Tension in the string.

therefore, $mg = T_{at\ eqn} = \frac{\lambda . b}{l}$ [λ = Hook's constant]

$$\Rightarrow \lambda = \frac{mgl}{b},$$



Let x be the displacement any time t, then the subsequent motion is governed by the following eqn

$$m\frac{d^2x}{dt^2} = mg - T^* = mg - \frac{\lambda(b+x)}{l}$$
$$= mg - \frac{mg}{b}(b+x)$$
$$= -\frac{mg}{b}x$$

or, $\frac{d^2x}{dt^2} = -g/b\, x$

Thus, the period of the free oscillation is $T_1 = \frac{2\pi}{\sqrt{g/b}}$,

For the forced oscillation, total vertical displacement becomes

$(l+b+x) + a\sin pt - l = b + x + a\sin pt = X$

therefore the eqn of motion is of the form

$$m\frac{d^2}{dt^2}(b+x+a\sin pt) = mg - T^* = mg - \lambda\frac{(b+x)}{l}$$

$$\Rightarrow m\left(\frac{d^2x}{dt^2} - ap^2\sin pt\right) = -\frac{mg}{b}x$$

or, $\frac{d^2x}{dt^2} = -g/b\, x + ap^2\sin pt$

or $\frac{d^2x}{dt^2} + n^2x = ap^2\sin pt$, where $g/b = n^2$.

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$\therefore \quad x(t) = c_1 \cos nt + c_2 \sin nt + \frac{ap^2 \sin pt}{D^2+n^2}$

$= c_1 \cos nt + c_2 \sin nt + \frac{ap^2 \sin pt}{-p^2+n^2}$ → Particular Integral

$(n \neq p)$

Initially at $t=0$, $x(t)=0$ & $\frac{dx}{dt} = -ap$, $\left[\frac{dx}{dt}\right|_{t=0} = \frac{dX}{dt} = \frac{d}{dt}(b+x+a\sin pt)\Big|_{t=0}$

$= 0 + \frac{dx}{dt} - ap\cos pt\Big|_{t=0}$

$= 0+0-ap.1$

$= -ap$

we have $c_1 = 0$ &

$-ap = c_2 n + \frac{ap^3}{n^2-p^2}$

$\Rightarrow c_2 = -\frac{1}{n} \cdot \frac{apn^2}{n^2-p^2} = -\frac{apn}{n^2-p^2}$.

Hence $x = -\frac{apn}{n^2-p^2}\sin nt + \frac{ap^2}{n^2-p^2}\sin pt$.

~~therefore~~, the total vertical distance is

$X = b + x + a\sin pt$

$= b - \frac{apn \sin nt}{n^2-p^2} + a \sin pt \left(\frac{ap^2}{n^2-p^2} + a\right)$

$= \left\{ b - \frac{apn}{n^2-p^2} \sin pt \right\} + \frac{an^2}{n^2-p^2} \sin pt.$

therefore, the contribution of the forced oscillation to the vertical displacement is $\frac{an^2 \sin pt}{n^2-p^2}$, where $T_1 = \frac{2\pi}{n}$ & $T_2 = \frac{2\pi}{p}$

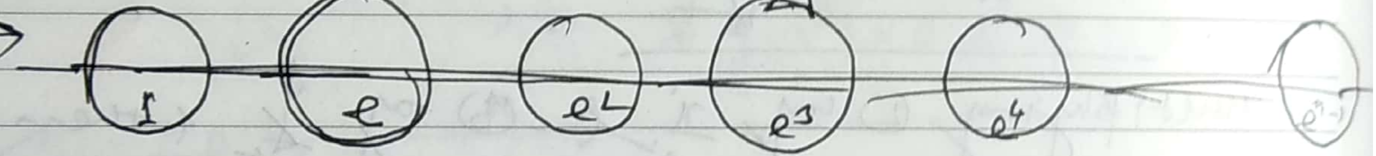
$= \frac{\left(\frac{2\pi}{T_1}\right)^2 \cdot a \sin pt}{\left(\frac{2\pi}{T_1}\right)^2 - \left(\frac{2\pi}{T_2}\right)^2}$

$= \frac{T_2^2}{T_2^2 - T_1^2} a \sin pt.$

Collision of Elastic Bodies

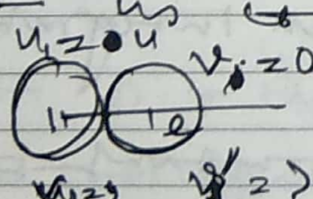
each (of elasticity e)

1. A series of n elastic bodies, spheres whose masses are $1, e, e^2, \ldots, e^{n-1}$ are at rest separated by intervals, with their centres on a straight line. The first is made to impinge directly on the second with ~~the 2nd~~ the velocity u. Show that finally the first (n-1) spheres will be moving with the same velocity $(1-e)u$ & the last one moves with the velocity u.



From the figure represents n spheres (elastic spheres) separated by intervals of masses $1, e, \ldots, e^{n-1}$ which are at rest with their centres on a straight line.

Let us consider the first two balls of masses 1 & e



[$u = u$ & $v = 0$ be the velocity of the balls before impact & v_1, v_1' be their velocity after impact

By Newton's law

$$v_1' - v_1 = e(u - 0)$$

$$v_1' - v_1 = eu \quad \text{——} (1)$$

From the conservation of linear momentum we have $1.u + e.0 = 1.v_1 + e v_1'$

$\therefore v_1 + e v_1' = u$ —— (ii)

Adding (i) & (ii) we have

$v'(1+e) = (e+1)u$

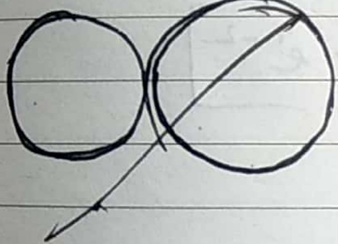
$\Rightarrow v' = u.$

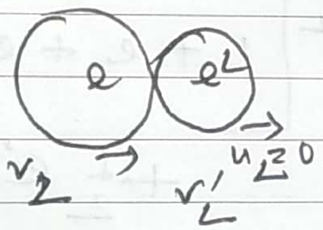
$\therefore v_1 = (1-e)u.$

Now after collision 2nd ball of mass e is move with velocity u & impinging the third ball of mass e^2 (at rest) with the vel. u.

If we consider the ~~first &~~ 2nd ball and 3rd ball

$u_1 = u$





Let v_2 & v_2' be the velocity of the 2nd & 3rd sphere just after impact

Then $v_2' - v_2 = e(u-0)$

$\Rightarrow v_2' - v_2 = eu$ —— (iii)

From the conservation of linear momentum we have

$e \cdot u + e^2 \cdot 0 = e v_2 + e^2 v_2'$ —— (iv)

From $\therefore v_2 + e v_2' = u$ —— (iv)

(iii) & (iv) we have

$(1+e) v_2' = (1+e)u$

$\therefore v_2' = u$ & $v_2 = (1-e)u.$

Proceeding in this way we have finally after impact the first $(n-1)$ ball of masses $1, e, \dots, e^{n-2}$ moves with the velocity $(1-e)u$ & the last ball of mass e^{n-1} moves with the velocity u along the line of contact

Therefore the ~~total~~ final K.E =

$$= \frac{1}{2}\cdot 1\cdot(1-e)^2u^2 + \frac{1}{2}e(1-e)^2u^2 + \dots + \frac{1}{2}e^{n-2}(1-e)^2u^2 + \frac{1}{2}e^{n-1}u^2$$

$$= \frac{1}{2}(1-e)^2u^2\left[1+e+e^2+\dots+e^{n-2}\right] + \frac{1}{2}e^{n-1}u^2$$

$$= \frac{1}{2}(1-e)^2u^2\cdot\frac{1-e^{n-1}}{1-e}u^2 + \frac{1}{2}e^{n-1}u^2$$

$$= \frac{1}{2}u^2\left[(1-e)(1-e^{n-1}) + e^{n-1}\right]$$

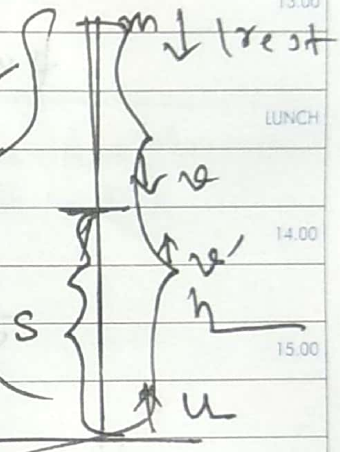
$$= \frac{1}{2}u^2\left[1-e-e^{n-1}+e^{n-1}+e^n\right]$$

$$= \frac{1}{2}(1-e+e^n)u^2$$

(18) An inelastic ball of mass m is dropped from a height h above the ground & at the same time a 2nd ball of mass M is projected vertically upwards to meet the former. Show that in order that immediately after collision, the balls may be at rest the 2nd ball must be project with a velocity

$$\sqrt{\left(\frac{m+M}{M}\right)gh}.$$

Sol. As the balls meet at a height s from the ground after time t and the 2nd ball of mass M is projected vertically with velocity u



Let $v\downarrow$ & $v'\uparrow$ (as shown in the figure) be the velocity of the balls of mass m & M just when they meet.

For the falling ball

$$v = gt \quad \text{---} (1)$$

$$v^2 = 2g(h-s) \quad \text{---} (2)$$

$$\text{where } h-s = \tfrac{1}{2}gt^2 \quad \text{---} (3)$$

For the ball rising

$$v'^2 = u^2 - 2gs \quad \text{---} (4)$$

$$s = ut - \tfrac{1}{2}gt^2 \quad \text{---} (5)$$

$$v' = u - gt \quad \text{---} (6)$$

9.00 From the conservation of linear momentum after impact both of them become motionless,

10.00 $$0 = mv - Mv'$$

$$\text{i.e. } \frac{v}{v'} = \frac{M}{m} \quad \text{—— (7)}$$

12.00 From eq (3) & (5) we have

$$h = ut \Rightarrow t = h/u \quad \text{—— (8)}$$

putting this value of t in eq (5) we have

$$s = h - \frac{1}{2} g \frac{h^2}{u^2}$$

From (2)

$$v^2 = 2gh - 2gs = 2g(h-s)$$

$$= 2g \cdot \frac{1}{2} g \frac{h^2}{u^2}$$

$$= \frac{g^2 h^2}{u^2}$$

& $v'^2 = u^2 - 2g\left(h - \frac{1}{2} g \frac{h^2}{u^2}\right)$

$$v'^2 = u^2 - 2gh + \frac{g^2h^2}{u^2} = \left(u - \frac{gh}{u}\right)^2$$

$$= \frac{(u^2 - gh)^2}{u^2}$$

$$u = \sqrt{\left(\frac{m+M}{M}\right) gh}$$

$$\frac{v^2}{v'^2} = \frac{M^2}{m^2} = \frac{g^2h^2}{(u^2-gh)^2}$$

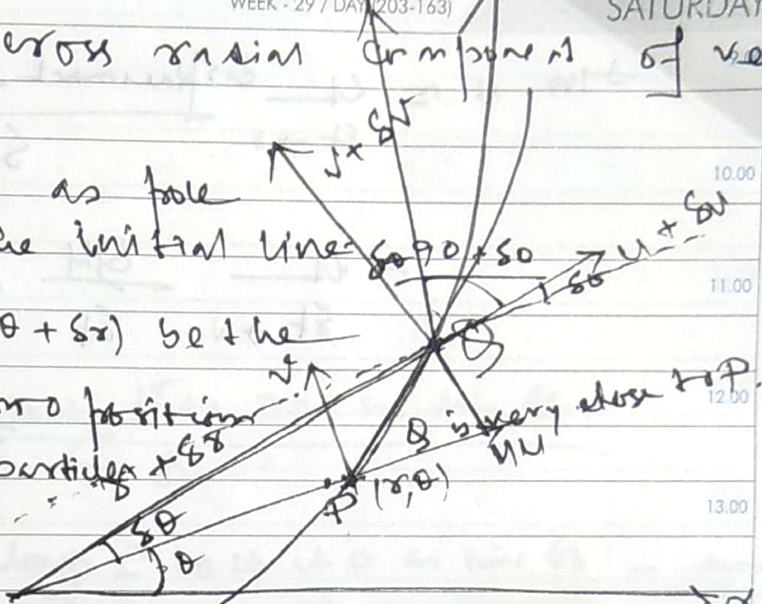
$$\frac{gh}{u^2 - gh} = \frac{M}{m} \quad \text{i.e. } \frac{u^2}{gh} = 1 + \frac{m}{M}$$

$$= \left(\frac{m+M}{M}\right)$$

Central Orbit & Planetary Motions

~~Radial & cross~~ Radial accⁿ.

Radial & transverse/cross radial component of velocity & accⁿ.



We take a fixed pt O as pole & the line OX as the initial line. Let (r, θ) & $(r+\delta r, \theta+\delta\theta)$ be the polar coordinates of two positions P & Q of a moving particle at time t & $t+\delta t$ on its path. Thus the chord PQ is the displacement of the particle in time δt. Let QM be the $\perp^r$

Here δt being very small, $\delta\theta$ is very small & δr is also very small.

Now $u = \lim_{\delta t \to 0} \frac{\text{displacement along OP in time } \delta t}{\delta t}$

$$= \lim_{\delta t \to 0} \frac{PM}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{OM - OP}{\delta t} = \lim_{\delta t \to 0} \frac{(r+\delta r)\cos\delta\theta - r}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(r+\delta r).1 - r}{\delta t},$$ since $\delta\theta \to 0$ as $\delta t \to 0$ & then $\delta\theta$ being very small radian measure, circular measure

$$= \lim_{\delta t \to 0} \frac{\delta r}{\delta t} = \frac{dr}{dt}.$$

Also $v = \lim_{\delta t \to 0} \dfrac{\text{displacement} \perp^r \text{ to OP at P in time } \delta t}{\delta t}$

$= \lim_{\delta t \to 0} \dfrac{QM}{\delta t} = \lim_{\delta t \to 0} \dfrac{(r+\delta r)\sin\delta\theta - 0}{\delta t}$

$= \lim_{\delta t \to 0} \dfrac{(r+\delta r)\cdot\delta\theta}{}$

$= \lim_{\delta t \to 0} \dfrac{r\,\delta\theta}{\delta t} = r\dot\theta$ | as $\delta t \to 0$ so $\delta\theta \to 0$

Thus $\dot r$ & $r\dot\theta$ are the radial & cross-radial velocity components of velocity of the particle. The actual velocity is the resultant of these two.

$v_r = u^2 + v^2 = \dot r^2 + (r\dot\theta)^2$.

Let f_1 & f_2 be the component of accⁿ of the particle along OP & $\perp^r$ to OP, then we have

$f_1 = \lim_{\delta t \to 0} \dfrac{\text{change of velocity along OP in time } \delta t}{\delta t}$

$= \lim_{\delta t \to 0} \dfrac{\text{velocity along OP in time } t+\delta - \text{velocity along OP}}{\delta t}$

$= \lim_{\delta t \to 0} \dfrac{\{(u+\delta u)\cos\delta\theta + (v+\delta v)\sin(90+\delta\theta)\} - u}{\delta t}$

$= \lim_{\delta t \to 0} \dfrac{\cancel{u} + \delta u - (v\,\delta\theta + \delta v\,\delta\theta) - \cancel{u}}{\delta t}$

as $\delta t \to 0$
$\cos\delta\theta \to 1$
$\sin\delta\theta \to \delta\theta$

9.00

$$= \frac{du}{dt} - v\frac{d\theta}{dt}$$

10.00

$$= \frac{d(\dot{r})}{dt} - r\dot{\theta}\frac{d\theta}{dt}$$

11.00

$$= \ddot{r} - r\dot{\theta}^2.$$

12.00

$$f_\perp = \underset{\delta t \to 0}{Lt} \frac{\text{change of velocity } \perp^r \text{ to OP in time } \delta t}{\delta t}$$

13.00

$$= \underset{\delta t \to 0}{Lt} \frac{\text{velocity along } \perp^r \text{ to OP at P in time } \delta t - \text{same as init}}{\delta t}$$

LUNCH

14.00

$$= \underset{\delta t \to 0}{Lt} \frac{(v+\delta v)\cos\delta\theta + (u+\delta u)\cos(90-\delta\theta) - v}{\delta t}$$

15.00

16.00

$$= \underset{\delta t \to 0}{Lt} \frac{v+\delta v + u\delta\theta + \delta u\delta\theta - v}{\delta t}$$

17.00

$$= \frac{dv}{dt} + u\frac{d\theta}{dt}$$

18.00

$$= \frac{d}{dt}(r\dot{\theta}) + \dot{r}\dot{\theta}$$

19.00

20.00

$$= 2\dot{r}\dot{\theta} + r\ddot{\theta}$$

21.00

$$= \frac{1}{r}\left[\frac{d}{dt}(r^2\dot{\theta})\right]$$

Notes

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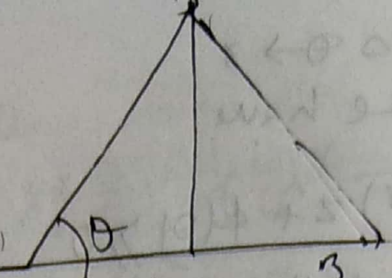
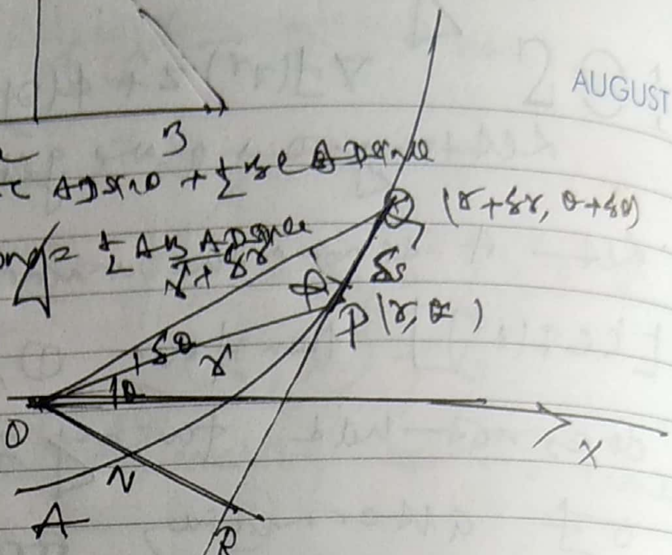


Short Note

Areal velocity

$\frac{1}{2} AC\cdot AD\sin\theta + \frac{1}{2} BC\cdot AD\sin\theta = \frac{1}{2} AB\cdot AD\sin\theta$



When particle moves along a plane curve, the rate of change of the area traced out by the radial vector joining the particle to the fixed pt. is called the areal velocity of the particle. This is actually rate of description of the sectorial area about O, the pole.

Let us consider $h = r^2\dot\theta = vp$ $\quad \frac{dA}{dt} = \frac{1}{2}h$

the point a particle moving along the curve APQ describes an arc PQ = δs in δt. Let the coordinate of P & Q are respectively (r,θ), $(r+\delta r, \theta+\delta\theta)$ referred to O, a fixed pt. as pole & OX, a fixed line, as initial line.

Then the areal velocity at P is

$$\lim_{\delta t\to 0}\frac{\Delta APQ}{\delta t} = \lim_{\delta t\to 0}\frac{\frac{1}{2} r(r+\delta r)\sin\delta\theta}{\delta t}$$

$$= \lim_{\delta t\to 0}\frac{\frac{1}{2} r^2\delta\theta}{\delta t}$$, neglecting small quantities above the first order

$$= \frac{1}{2} r\frac{d\theta}{dt} = \frac{1}{2} r^2\dot\theta = \frac{1}{2}h$$

Alternatively,

$$\lim_{\delta t\to 0}\frac{\Delta OPQ}{\delta t} = \lim_{\delta t\to 0}\frac{\frac{1}{2} PQ\, p}{\delta t} = \lim_{\delta t\to 0}\frac{\frac{1}{2}\delta s\, p}{\delta t}$$

$$= p\lim_{\delta t\to 0}\frac{\delta s}{\delta t} = p\frac{ds}{dt} = pv$$

Hence areal velocity $2A = h = vp$

$r = a e^{\theta \cot\alpha}$

$$\frac{dr}{dt} = a e^{\theta\cot\alpha} \cdot \cot\alpha \frac{d\theta}{dt}$$

$$= r \cot\alpha \frac{d\theta}{dt}$$

$$\frac{d^2 r}{dt^2} = \cot\alpha\left[r\frac{d^2\theta}{dt^2} + \frac{dr}{dt}\frac{d\theta}{dt}\right]$$

$$= \cot\alpha\left[r\frac{d^2\theta}{dt^2} + r\cot\alpha\left(\frac{d\theta}{dt}\right)^2\right]$$

$$\Rightarrow \frac{d^2 r}{dt^2} = r\cot\alpha\frac{d^2\theta}{dt^2} + r\cot^2\alpha\left(\frac{d\theta}{dt}\right)^2$$

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = r\cot\alpha\frac{d^2\theta}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 + r\cot^2\alpha\left(\frac{d\theta}{dt}\right)^2$$

$v\frac{dv}{dr} = -F$

$\int_0^v v\,dv = -\int_a^r F\,dr =$

$v^2 = 2F\alpha$

$\int_0^v v\,dv = -\int_\infty^r \frac{\mu}{r^2}\,dr$

$\frac{1}{r} = \frac{1}{a}$

$\operatorname{cosec}^2\theta - \cot^2\theta = 1$

$v^2 = \frac{2\mu}{r}$

Differentiation eqn of a central orbit in a polar coordinate system

Let P be the position of the particle at time t. Let (r, θ) be the coordinates of P referred to the fixed pt. O as the fixed initial axis OX.

Let F be the force per unit mass, then mF be the ~~actual~~ attracting force acting on it. Hence the eqn of motion in the radial & cross-radial directions are

$$m(\ddot{r} - r\dot{\theta}^2) = -mF \quad \text{——— (1)}$$

$$\& \quad \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0 \quad \text{——— (2)}$$

integrating (2) we have

$$r^2\dot{\theta} = \text{constant} = h \text{ (say)}$$

$$\text{or} \quad \frac{d\theta}{dt} = hu^2 \quad \text{——— (3)}$$

$ur = 1$

Again $\dot{r} = \frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{u}\right) = \frac{d}{d\theta}\left(\frac{1}{u}\right)\frac{d\theta}{dt}$

$= -\frac{1}{u^2}\frac{du}{d\theta}\cdot\dot{\theta}$

$= -h\frac{du}{d\theta}$ — (1)

$\ddot{r} = \frac{d}{dt}\left(-h\frac{du}{d\theta}\right)$

$= -h\frac{d^2u}{d\theta^2}\cdot\frac{d\theta}{dt}$

$= -h\frac{d^2u}{d\theta^2}\cdot hu^2$

$= -h^2u^2\frac{d^2u}{d\theta^2}$ — (2)

$r \cdot h^2u^2$

Substituting this value in (1) we get

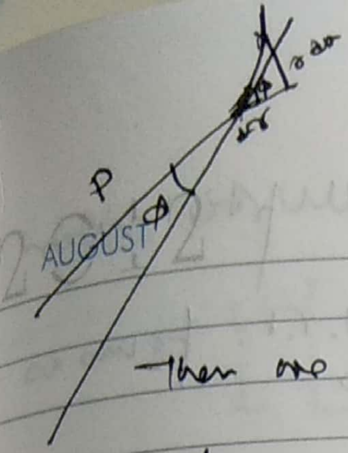
$-h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3 = -F$

or $\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2}$ — (3)

which is the differential eqn of the path where F being the force of attraction.

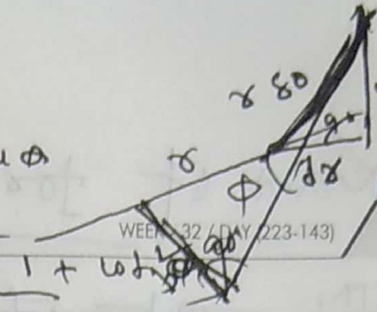
Differential eqn in pedal coordinate

Let p be the length of the perpendicular from the fixed centre of Force upon the tangent to the tangent at P whose polar coordinates is (r, θ)



$p = r\sin\phi$

$\frac{1}{p} = \frac{1}{r}\operatorname{cosec}\phi$



$\frac{1}{p^2} = \frac{1}{r^2}\left[1 + \cot^2\phi\right]$

Then we have $= \frac{1}{r^2}\left[1 + \left(\frac{1}{r}\frac{dr}{d\theta}\right)^2\right]$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

~~Differentiating~~ w.r.t. r we have

$$-\frac{2}{p^3}\frac{dp}{dr} \cancel{= -\frac{2}{r^3} - \frac{4}{r^5} \cdot 2\frac{dr}{d\theta}\frac{d^2r}{d\theta^2}\frac{d\theta}{dr}}$$

or $\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$, where $u = \frac{1}{r}$ — (1)

Differentiating (1) w.r.t. r we have

$$-\frac{2}{p^3}\frac{dp}{dr} = \frac{d}{du}\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]\frac{du}{dr}$$

$$= \left[2u + 2\cancel{\frac{du}{d\theta}}\frac{d^2u}{d\theta^2}\cancel{\frac{d\theta}{du}}\right]\frac{du}{dr}$$

$$= 2\left[u + \frac{d^2u}{d\theta^2}\right]\left(-\frac{1}{r^2}\right)$$

or $\frac{r^2}{p^3}\frac{dp}{dr} = u + \frac{d^2u}{d\theta^2} = \frac{F}{h^2u^2}$

Notes

$$\therefore \boxed{F = \frac{h^2}{p^3}\frac{dp}{dr}}$$

11 11.08.2012 SATURDAY WEEK 32 / DAY (224-142) AUGUST

Law of force for an elliptic motion

The eq of an ellipse w.r.t. focus as pole is

$$\frac{l}{r} = 1 + e\cos\theta$$

or

$$u = \frac{1}{l}\left[1 + e\cos\theta\right]$$

$$\therefore \frac{du}{d\theta} = -\frac{e\sin\theta}{l}$$

$$\frac{d^2u}{d\theta^2} = -\frac{e\cos\theta}{l}$$

Hence $F = h^2u^2\left[u + \frac{d^2u}{d\theta^2}\right]$

$$= \frac{h^2}{r^2}\left[\frac{1}{l}\left[1 + \cancel{e\cos\theta} - \cancel{e\cos\theta}\right]\right]$$

$$= \frac{h^2}{l} \cdot \frac{1}{r^2} = \frac{\mu}{r^2} \text{ where } \mu = \frac{h^2}{l}$$

12 SUNDAY

Thus the central force varies inversely as the square of the radial distance.

Again $v^2 = \frac{h^2}{p^2} = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]$

$$= \frac{h^2}{l^2}\left[(1 + e\cos\theta)^2 + (e\sin\theta)^2\right]$$

$$= \frac{h^2}{l^2}\left[1 + 2e\cos\theta + e^2\right]$$

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Notes

$$= \frac{h^2}{l}\left[\frac{2(1+e\cos\theta)}{l} - \frac{1-e^2}{l}\right]$$

$$= \mu\left[\frac{2}{r} - \frac{b^2/a^2}{b^2/a}\right]$$

$$= \mu\left[\frac{2}{r} - \frac{1}{a}\right]$$

$\left(\frac{a}{r}\right)^n = \cos n\theta$

$n = 3/2$

$\left(\frac{a}{r}\right)^n = \cos n\theta$

where $n = 3/2$

Apses: An apse is a point on a central orbit at which the radial vector drawn from the centre is either maximum / minimum i.e. $\frac{dr}{d\theta} = 0$ / $\frac{du}{d\theta} = 0$

The length of the radial vector corresponding to such pt. is known as apsidal distance.

A particle moving under a constant force from the centre is projected in a dirn ⊥r to the radial dirn with th velocity acquired in falling to the pt. of projection from centre ~~show~~ that the path is $\left(\frac{a}{r}\right)^3 = \cos^2\frac{3}{2}\theta$.

$\left[\left(\frac{a}{r}\right)^{3/2} = \cos\frac{3}{2}\theta\right]$

~~Sol~~ ∴ the force is constant F (say)

The differential eq of the path

$$\frac{d^2u}{d\theta^2} + u = -\frac{F}{h^2u^2}$$

If v be the velocity of projection at a distance a from the centre, then

$$v\frac{dv}{d\theta} = F \qquad \int_0^v v\,dv = \int_0^a F\,dr = Fa$$

$$\frac{1}{2}v^2 = Fa$$

Therefore $h^2 = v^2p^2 = v^2a^2 = 2Fa^3$

$$\therefore F/h^2 = \frac{1}{2a^3}$$

Hence

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{2a^3u^2}$$

Multiplying both sides by $2u\frac{du}{d\theta}$ & integrating we get

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{1}{a^3u} + C$$

Now at $u = 1/a$, $\frac{du}{d\theta} = 0 \Rightarrow C = 0$

$$\left(\frac{du}{d\theta}\right)^2 = \frac{1}{a^3u} - u^2 = \frac{1-a^3u^3}{a^3u}$$

$$\pm d\theta = \sqrt{\frac{a^3u}{1-(au)^3}}\,du = \sqrt{\frac{au}{1-(au)^3}}\,a\,du$$

put $(au)^3 = \cos^2\phi$

$$au = \cos^{2/3}\phi$$

$$a\,du = -\frac{2}{3}\cos^{-1/3}\phi \cdot \sin\phi\,d\phi$$

$$\pm d\theta = \sqrt{\frac{a^2\cos^{2/3}\phi}{\sin^2\phi}} \cdot \frac{2}{3a}\frac{\sin\phi\,d\phi}{\cos^{1/3}\phi}$$

$$\pm d\theta = \frac{2}{3}\,d\phi$$

$\Rightarrow \phi = 3/2\,\theta$. since ϕ & θ vanishes simultaneously

Hence $(au)^3 = \cos^2 \frac{3}{2}\theta$

$\Rightarrow \frac{a}{r} = \cos^{4/3} \frac{3}{2}\theta$

$\Rightarrow \left(\frac{a}{r}\right)^3 = \cos^4 \frac{3}{2}\theta$

15.08.2012

WEEK - 33 / DAY (228-138) WEDNESDAY

9.00

10.00

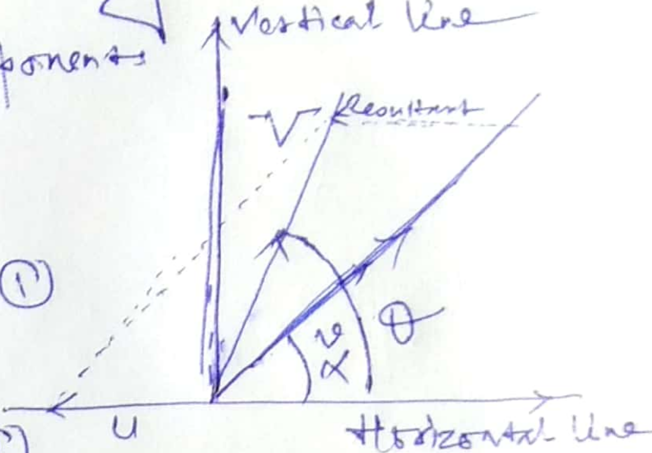
11.00

... 27 [Momental ellipsoid]

Problem 2. A shell of mass m is fired from a gun of mass M, which can recoil freely on the horizontal base and the elevation of the gun is α. Prove that the inclination of the path of the shell to the horizontal line will be $\tan^{-1}\left(\left(1+\frac{m}{M}\right)\tan\alpha\right)$.

Sol:

Let v be the velocity of the shell relative to the barrel of the gun. Let u be the velocity of recoil of the gun and V be the actual velocity of the shell.



The horizontal & vertical components are given by

$$V\cos\theta = v\cos\alpha - u\cdot\cos 0 = v\cos\alpha - u \quad \text{—— (i)}$$

$$V\sin\theta = v\sin\alpha - u\sin 0 = v\sin\alpha - 0 = v\sin\alpha \quad \text{—— (ii)}$$

The horizontal component of the impulse is responsible for backward momentum of the gun.

[Momentum is taken over actual / resultant velocity]

Therefore,

$$mV\cos\theta + (-Mu) = (m+M)\cdot 0$$

or, $mV\cos\theta = Mu$ —— (iii)

From (i) & (ii), we have

$$\tan\theta = \frac{v\sin\alpha}{v\cos\alpha - u}$$

$$= \frac{mv\sin\alpha}{m(v\cos\alpha - u)}$$

$$= \frac{mv\sin\alpha}{mV\cos\theta} = \frac{(mv\cos\alpha)\tan\alpha}{Mu}$$

$$= \frac{(mV\cos\theta + mu)\tan\alpha}{Mu} \quad [\text{by (i)}]$$

$$= \frac{(Mu + mu)\tan\alpha}{Mu} \quad [\text{by (iii)}]$$

$$= \left(1+\frac{m}{M}\right)\tan\alpha$$

or, $\theta = \tan^{-1}\left(\left(1+\frac{m}{M}\right)\tan\alpha\right)$

= actual inclination of the shell.

JUNE 18.06.2012 18 MONDAY

WEEK - 25 / DAY (170-196)

Problem 3.(B) ~~Prove that~~ the kinetic energy of two particle of masses m & m' moving in a plane is $\frac{1}{2}(m+m')V^2 + \frac{1}{2}\frac{mm'}{m+m'}v^2$

where V is the velocity of the centre of mass & v is the velocity either of them w.r.t. other.

Sol. ~~Let the~~ particle of mass m & m' move in the xy plane. At any instant t P be the position of mass m & Q be the ~~posn~~ of the mass m' & R be the position of the centre of mass. ~~Let~~ further v_1, v_2, V be the respective velocity.

If (v_1, v_1'), (v_2, v_2') & (V, V') be their respective horizontal & vertical velocities,

Then we have
$$mv_1 + m'v_2 = (m+m')u_1 \quad \text{—(i)}$$
$$mv_1' + m'v_2' = (m+m')u_2 \quad \text{—(ii)}$$
$$u_1^2 + u_2^2 = V^2 \quad \text{— — — (iii)}$$
&
$$(v_1 - v_2)^2 + (v_1' - v_2')^2 = v^2 \quad \text{— — (iv)}$$

From the first two eq we have
$$(mv_1 + m'v_2)^2 + (mv_1' + m'v_2')^2 = (m+m')^2V^2 \quad \text{(v) (by (iii))}$$

multiplying both sides of eq (iv) by mm' & then adding with (v) we have

$$m^2v_1^2 + m'^2v_2^2 + 2mm'v_1v_2 + m^2v_1'^2 + m'^2v_2'^2 + 2mm'v_1'v_2'$$
$$+ mm'v_1^2 + mm'v_2^2 - 2mm'v_1v_2 + mm'v_1'^2 + mm'v_2'^2 - 2mm'v_1'v_2'$$
$$= mm'v^2 + (m+m')^2V^2$$

$$m(m+m')v_1^2 + m'(m+m')v_2^2 + m(m+m')v_1'^2 + m'(m+m')v_2'^2$$

Notes

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$$\therefore \; m(m+m')(v_1^2+v_1'^2) + m'(m+m')(v_2^2+v_2'^2) = (m+m')^2 v^2 + mm' v_r^2$$

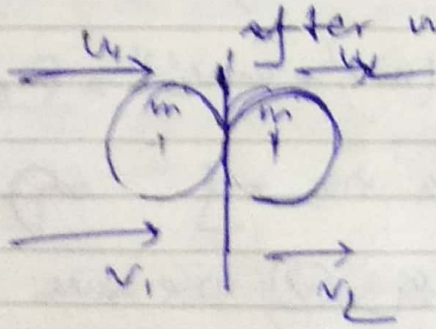
$$\therefore \; \frac{1}{2} m v_1^2 + \frac{1}{2} m' v_2^2 = \frac{1}{2}(m+m') v^2 + \frac{1}{2}\frac{mm'}{m+m'} v_r^2.$$

Newton's experimental law:

When two bodies impinge directly, their relative velocity after impact is in a constant ratio to their relative velocity before impact & is in the opposite direction.

$$v_1 - v_2 = -e(u_1 - u_2) \qquad v_r \text{ (after impact)} = -e(\text{bef}$$

(25) Two elastic spheres, each of mass m, collide directly. Show that the energy loss during the impact is $\frac{1}{4}m(u^2-v^2)$, where u & v are their relative velocities before & ~~after~~ after impact.



Sol: u_1, u_2 be the velocities of the spheres just before impact & v_1, v_2 be their velocities just after impact.

By question: $u_1 - u_2 = u$
& $v_2 - v_1 = v$ } ; (A)

By the Newton's experimental law

$v_2 - v_1 = e(u_1 - u_2)$ —— (B)

From the conservation of linear momentum we have $mu_1 + mu_2 = mv_1 + mv_2$

$\Rightarrow u_1 + u_2 = v_1 + v_2$ —— (C)

the loss of K.E. $= \frac{1}{2}mu_1^2 + \frac{1}{2}mu_2^2 - \frac{1}{2}mv_1^2 - \frac{1}{2}mv_2^2$

$= \frac{1}{2}m\left[(u_1^2 + u_2^2) - (v_1^2 + v_2^2)\right]$

$= \frac{1}{2}m\left[\left[(u_1 - u_2)^2 + 2u_1u_2\right] - \left[(v_1 - v_2)^2 + 2v_1v_2\right]\right]$

$= \frac{1}{4}m\left[2(u^2 - v^2) + 4(u_1u_2 - v_1v_2)\right]$

$= \frac{1}{4}m\left[2u^2 - 2v^2 - u^2 + v^2\right]$

Now $4u_1u_2 = (u_1 + u_2)^2 - (u_1 - u_2)^2$

$4v_1v_2 = (v_1 + v_2)^2 - (v_1 - v_2)^2$ $= \frac{1}{4}m(u^2 - v^2)$

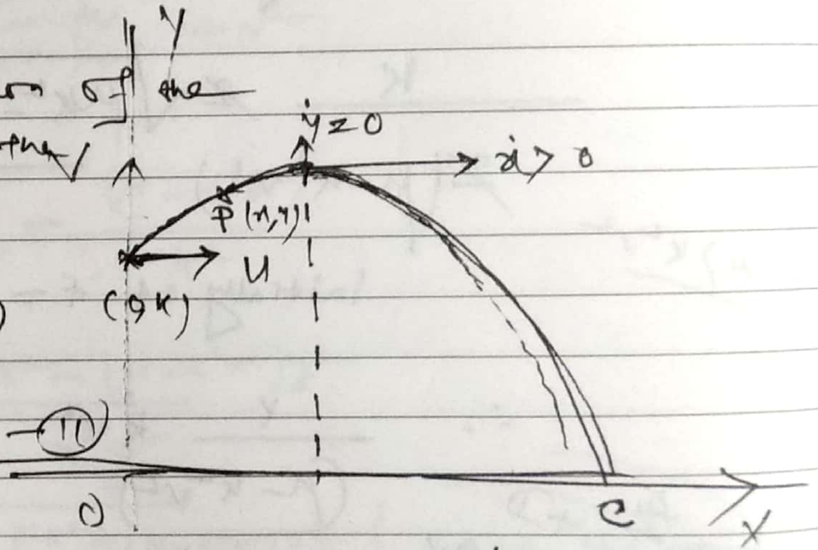
$\therefore 4(u_1u_2 - v_1v_2) = \left(\cancel{(u_1 + u_2)^2} - \cancel{(v_1 + v_2)^2}\right) - u^2 + v^2$

$= v^2 - u^2$

A particle is moving under the influence of an attractive force $m\mu/y^3$ towards the axis of x. Show that, if it be projected from a pt. $(0,k)$ with the component velocity u & v parallel to the axis of x & y respectively. Prove that it will not strike the axis of x unless $\mu > v^2k^2$ and in this case the distance of the pt. of impact from the origin is $\left(\frac{ukL}{\sqrt{\mu}-vk}\right)$.

Sol →

Let $P(x,y)$ be the position of the particle at any time t. Then the eq of motion are



$$\frac{d^2x}{dt^2} = 0 \quad \text{——(1)}$$

$$\frac{d^2y}{dt^2} = -\frac{\mu}{y^3} \quad \text{——(2)}$$

From (1) integrating twice one by one we have

$x(t) = c_1 t + c_2$, where c_1, c_2 are arbitrary constants

Initially at $t=0$, $x=0 \Rightarrow c_2 = 0$

& at $t=0$, $\dot{x}=u \Rightarrow c_1 = u$

$\therefore x(t) = ut$ ——— (3)

Multiplying (2) by $2\dot{y}$ both sides and integrating we have

$$\dot{y}^2 = \frac{\mu}{y^2} + d_1$$

initially at $y=k$, $\dot{y}=v$

$\Rightarrow v^2 = +\frac{\mu}{k^2} + d_1$

$\alpha = v^2 - \mu/k^2$

Therefore $\dot{y}^2 = \mu/y^2 + \underline{(v^2 - \mu/k^2)}$ — (4)

From eq (4), we have seen that the particle strikes the axis of x only when $\dot{y}$ decreases & ultimately it becomes zero, ie $\dot{y} = 0$, which is possible only when the term μ/y^2 is balanced by $v^2 - \mu/k^2$ ie $v^2 - \mu/k^2$ must be negative.

Hence the particle strikes the ground when

$v^2 - \mu/k^2 < 0$ ie $\underline{\mu > v^2k^2}$ — (5)

or $\dot{y}^2 = \mu/y^2 - \frac{(\mu - v^2k^2)}{k^2}$ $\quad (\mu - v^2k^2 > 0)$

From the figure, we have $\dot{y} = 0$ at $y = b$ (say)

Therefore, $0 = \mu/b^2 - \frac{(\mu - v^2k^2)}{k^2}$

ie $\mu/b^2 = \frac{\mu - v^2k^2}{k^2}$

or $b = \frac{\sqrt{\mu}\, k}{\sqrt{\mu - v^2k^2}}$ — (6)

From eq (4), we have

$\dot{y}^2 = \mu/y^2 - \mu/b^2 = \frac{\mu(b^2 - y^2)}{b^2y^2}$

From the motion of the particle from A to B, we have $\dot{y} = + \frac{\sqrt{\mu}}{b} \frac{\sqrt{b^2 - y^2}}{y}$; as $\dot{y} > 0$ in between A to B.

$b = \cancel{\dfrac{\sqrt{\mu}\, v}{\sqrt{\mu - v^2 k^2}}}$

Let the particle takes time t_1 to travell the distance AB, then

$$\int_0^{t_1} dt = + \int_k^b \frac{\sqrt{b}}{\sqrt{\mu}} \frac{y\, dy}{\sqrt{b^2-y^2}}$$

$$= -\frac{b}{2\sqrt{\mu}} \int_k^b \frac{d(b^2-y^2)}{(b^2-y^2)^{1/2}}$$

$$= -\frac{b}{2\sqrt{\mu}} \frac{\sqrt{(b^2-y^2)}}{1/2} \Bigg|_k^b$$

$$= \frac{b}{\sqrt{\mu}} \sqrt{b^2-k^2}$$

For the motion of the particle from B to C, we have

$$\frac{dy}{dt} = -\frac{\sqrt{\mu}}{b} \frac{\sqrt{b^2-y^2}}{y} \quad (\text{From } B \text{ to } C,\ y \text{ is -ve})$$

or $dt = \dfrac{-b}{\sqrt{\mu}} \dfrac{y}{\sqrt{b^2-y^2}}$

If t_2 be the time taken to move from B to C, then

$$\int_0^{t_2} dt = \frac{-b}{\sqrt{\mu}} \int_b^0 \frac{y\, dy}{\sqrt{b^2-y^2}}$$

$$= \frac{b}{\sqrt{\mu}} \sqrt{b^2-y^2} \Bigg|_b^0$$

$$= \frac{b^2}{\sqrt{\mu}}.$$

Total time required to travell the distance between A & B is

$$t_1+t_2 = \frac{b}{\sqrt{\mu}}\sqrt{b^2-k^2} + \frac{b^2}{\sqrt{\mu}}$$

$$= \frac{b}{\sqrt{\mu}}\left(\sqrt{b^2-k^2}+b\right)$$

$$= \frac{k}{\sqrt{\mu - v^2k^2}}\left[\frac{vk^2}{\sqrt{\mu-v^2k^2}} + \frac{\sqrt{\mu}\,k}{\sqrt{\mu - v^2k^2}}\right]$$

$$= \frac{k^2\left(vk+\sqrt{\mu}\right)}{\left(\mu - v^2k^2\right)} = \frac{k^2}{\sqrt{\mu}-vk}$$

Hence the total distance of the point of impact from the origin is $v(t_1+t_2)$

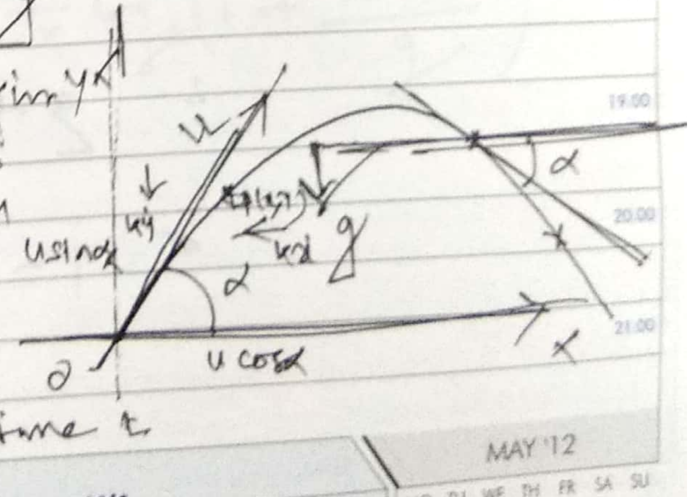
$$= \frac{vk^2}{\sqrt{\mu}-vk}$$

Problem :

5. A particle is projected with a velocity u at an inclination α above the horizontal line in a medium whose resistance per unit mass is k times the velocity. Show that its direction will again make an angle α below the horizontal line after a time

$$\frac{1}{k}\log\left(1+\frac{2ku}{g}\sin\alpha\right).$$

Let us assume O the origin as the pt. of projection and the horizontal & vertical line through O as the axes.



Let P(x,y) be the position of the particle at any time t.

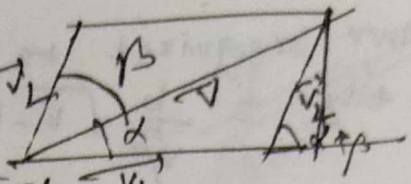
Then the eqn of motion are

$$\frac{d^2x}{dt^2} = -k\dot x \quad \text{— (1)}$$

$$\& \quad \frac{d^2y}{dt^2} = -g - k\dot y \quad \text{— (2)}$$

$\{v_1 + v_2\cos(\alpha+\beta)\}^2 + (v_2\sin(\alpha+\beta))^2 = v^2$

$v_1^2 + v_2^2 + 2v_1v_2\cos(\alpha+\beta) = v^2$



$v\cos\alpha = v_1 + v_2\cos(\alpha+\beta)$

$v\sin\alpha = v_2\sin(\alpha+\beta)$

From ① we have

$$\frac{d\dot{x}}{\dot{x}} = -k\,dt + \log A$$

or $\dot{x} = Ae^{-kt}$, initially at $t = 0$, $\dot{x} = u\cos\alpha$

$\Rightarrow A = u\cos\alpha$

or $$\frac{dx}{dt} = u\cos\alpha\, e^{-kt} \qquad \text{——— ③}$$

From ②, we have

$$\frac{k\,d\dot{y}}{g + k\dot{y}} = -k\,dt$$

on integrating, we have

$$\ln(g + k\dot{y}) = -kt + B$$

initially at $t = 0$, $\dot{y} = u\sin\alpha$

$\therefore \ln(g + ku\sin\alpha) = B$

$\therefore g + k\dot{y} = (g + ku\sin\alpha)e^{-kt}$

or $\dot{y} = -\frac{g}{k} + \left(\frac{g}{k} + u\sin\alpha\right)e^{-kt}$.

$$\left(x - \frac{uv}{g}\right)^2 = -4 \cdot \frac{u^2}{2g}\left(y - \frac{v^2}{2g}\right)$$

If the direction of motion makes an angle ψ with the horizontal line at any time t, then

$$\tan\psi = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\left(u\sin\alpha + \frac{g}{k}\right)e^{-kt} - \frac{g}{k}}{u\cos\alpha\, e^{-kt}}$$

$$= \tan\alpha + \frac{g}{ku\cos\alpha} - \frac{g}{ku\cos\alpha}e^{kt}$$

If the direction of motion again makes an angle α (3) below the horizontal after a time T, then $\psi = -\alpha$ at $t = T$,

Therefore

$$\tan(-\alpha) = \tan\alpha + \frac{g}{ku\cos\alpha} - \frac{g}{ku\cos\alpha}e^{kT}$$

$$\text{or,}\quad \frac{g}{\cancel{ku\cos\alpha}}e^{kT} = 2\tan\alpha + \frac{g}{ku\cos\alpha}$$

$$= \frac{2ku\sin\alpha + g}{\cancel{ku\cos\alpha}}$$

$$\text{or}\quad e^{kT} = \left(1 + \frac{2ku}{g}\sin\alpha\right)$$

$$T = \frac{1}{k}\log_e\left(1 + \frac{2ku\sin\alpha}{g}\right).$$

PLANETARY MOTION

Orbit described under inverse square law

Let the acceleration be always directed towards a fixed point and be equal $\frac{\mu}{(\text{distance})^2}$.

Thus we have $\quad F = \frac{\mu}{r^2} \qquad \text{---} (1)$

where F is the acceleration at a distance r from the centre of force.

Then the diff. eqn of the path in pedal form is

$$\frac{h^2}{p^3}\cdot\frac{dp}{dr} = \frac{\mu}{r^2}$$

$$\Rightarrow -\frac{2h^2}{p^3}\,dp = -\frac{2\mu}{r^2}\,dr$$

Integrating, we get

$$v^2 = \frac{h^2}{p^2} = \frac{2\mu}{r} + c, \quad \text{where } c \text{ is constant} \qquad \text{---} (ii)$$

Now the pedal eqns of an ellipse and hyperbola are

$$\frac{b^2}{p^2} = \frac{2a}{r} - 1 \quad \text{and} \quad \frac{b^2}{p^2} = \frac{2a}{r} + 1 \qquad \text{---} (iii)$$

respectively, where a and b are semi-transverse and semi conjugate axes respectively. In each case, the focus of the curve is the pole.

Comparing (ii) with (iii), we see that

If $c < 0$ ie $v^2 < \frac{2\mu}{r}$, then the orbit is an ellipse.

If $c > 0$ ie $v^2 > \frac{2\mu}{r}$, then the orbit is a hyperbola.

If $c = 0$ ie $v^2 = \frac{h^2}{p^2} = \frac{2\mu}{r}$, then the orbit is a parabola.

In the case of an ellipse, comparing (ii) and (iii), we have

$$\frac{h^2}{b^2} = \frac{\mu}{a} = -\frac{c}{1}$$

$$\Rightarrow \quad h = \sqrt{\mu\frac{b^2}{a}} = \sqrt{\mu \times \text{semi-latus rectum}}$$

and $c = -\frac{\mu}{a}$. Thus for an ellipse

In the case of a hyperbola, comparing (ii) and (iii), we get

$$\frac{h^2}{b^2} = \frac{\mu}{a} = \frac{c}{1}$$

Therefore, $h = \sqrt{\mu \frac{b^2}{a}} = \sqrt{\mu \times \text{semi-latus rectum}}$,

and $c = \frac{\mu}{a}$

Hence in this case $v^2 = \mu\left(\frac{2}{r} + \frac{1}{a}\right)$

Again for parabola, we get

$$v^2 = \frac{2\mu}{r}$$

Corollary - 1

The orbit under inverse square of the distance is an ellipse, a parabola or a hyperbola according as the velocity at any point is less than equal to or greater than that acquired in falling from infinity to that point.

We know that if a particle be projected at a distance R with a velocity v in any direction under inverse square of the distance, then the path is an ellipse, a parabola or a hyperbola according as

$$v^2 < = \text{ or } > \frac{2\mu}{R}$$

Now, the law of force being inverse square, the equation of motion is

$$\frac{d^2 r}{dt} = -\frac{\mu}{r^2}$$

the acceleration being towards the centre of force

Integrating, we get

$$\dot{r}^2 = \left(\frac{dr}{dt}\right)^2 = \left[\frac{2\mu}{r}\right]_{\infty}^{R} = \frac{2\mu}{R}$$

Hence the corollary is proved.

Corollary 2

periodic time for an elliptic orbit.

$\Rightarrow$ We know that $\frac{1}{2}h$ is the areal velocity and if T be the periodic time for an elliptic orbit, then we have

$$T = \frac{\text{area of the ellipse}}{\frac{1}{2}h} = \frac{\pi ab}{\frac{1}{2}\sqrt{\mu \frac{b^2}{a}}} = \frac{2\pi}{\sqrt{\mu}} a^{3/2}$$

Thus the square of the periodic time for an elliptic orbit described under inverse square of the distance varies as the cube of the semi-major axis of the ellipse.

Corollary 3

Let V_1 be the velocity for the description of a circle of radius R.

Then $\frac{V_1^2}{R}$ = normal acceleration $= \frac{\mu}{R^2}$

Therefore, $V_1^2 = \frac{\mu}{R}$

$$V_1 = \frac{\text{velocity from infinity}}{\sqrt{2}}$$

Planetary orbit in polar equation

We can find the polar eqn of the orbit described under inverse square law of force.

Here $F = \frac{\mu}{r^2} = \mu u^2$ — (1)

The differential eqn of the path is then

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}$$

$$\Rightarrow \frac{d^2}{d\theta^2}\left(u - \frac{\mu}{h^2}\right) + \left(u - \frac{\mu}{h^2}\right) = 0$$

put $U = u - \frac{\mu}{h^2}$, so that the eqn becomes

$$\frac{d^2U}{d\theta^2} + U = 0$$

Then general soln of this eqn is $U = A\cos(\theta - \epsilon)$

or $u = \frac{\mu}{h^2} + A\cos(\theta - \epsilon)$ — (ii)

where A and ϵ arbitrary constants.

(ii) may put in the form, $\frac{h^2/\mu}{r} = 1 + \frac{Ah^2}{\mu}\cos(\theta - \epsilon)$

Now the eqn of the conic with focus as pole as

$$\frac{l}{r} = 1 \pm e\cos\theta,$$

where l is the semi latus rectum and e is the eccentricity.
Comparing (iii) and (iv), the path described is a conic whose semi latus rectum is $\frac{h^2}{\mu}$ and eccentricity is $\left(\pm \frac{Ah^2}{\mu} \right)$.

Kepler's laws of planetary motion

(i) Each planet describes an ellipse having the sun at one of its foci.

(ii) The radius vector drawn from the sun to the planet describes equal areas in equal times.

(iii) The square of the periodic time of a planet is proportional to the cube of the semi-major axis of its orbit.

Deductions from Kepler's law

From the 2nd law, we see that the rate of description of sectorial area by the planet about the sun is constant. Hence $r^2\dot\theta$ is constant so that the cross-radial acceleration of the central orbit which is $\frac{1}{r}\frac{d}{dt}(r^2\dot\theta)$ is zero.

Thus the planet has only radial acceleration towards the sun. From the 1st law, we see that the path of a planet is elliptic about the sun. This combined with the previous result, clearly shows that the law of force under which a planet moves is the inverse square law of distance, the force being directed to the focus. Thus if the law of force be $\frac{\mu}{r^2}$, then

$$h^2 = \mu l = \mu \frac{b^2}{a} = \mu a(1-e^2)$$

where l is the semi-latus rectum, e is the eccentricity and a, b are the semi-major and semi-minor axes respectively.

Again, the path being elliptic, $v^2 = \frac{h^2}{p^2} = \frac{h^2}{l^2}\cdot\frac{l}{p^2} = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$.

where p is the length of the perpendicular from the pole upon the tangent at a point on the ellipse at a distance r and v is the velocity of the planet there.

If T be the periodic time of the planet, then

$$h = 2r^2\dot\theta = 2\cdot(\text{areal velocity}) \text{ and } Th = 2\pi ab$$

Therefore, $T = \frac{2\pi ab}{h} = \frac{2\pi ab}{\sqrt{\mu l}} = \frac{2\pi}{\sqrt{\mu}} a^{3/2}$. [Complete]

ANALYTICAL DYNAMICS OF A PARTICLE

— S. GANGULY & S. SAHA.

Motion under inverse square & planetary orbits

Exercises - 6 (I).

(4) A particle describes an ellipse under a force $\frac{\mu}{(\text{distance})^2}$ towards the focus, if it was projected with velocity $\sqrt{2}V$ from a point distant r from the centre of force, show that the periodic time is $\frac{\pi}{\sqrt{2\mu}}\left[\frac{1}{r}-\frac{V^2}{\mu}\right]^{-\frac{3}{2}}$.

Soln:
For an elliptic orbit, we have

$$v^2 = \mu\left(\frac{2}{r}-\frac{1}{a}\right)$$

where a is the semi transverse axis of the ellipse.

At a distance r, $v=\sqrt{2}V$

Then,
$$2V^2 = \mu\left(\frac{2}{r}-\frac{1}{a}\right)$$

$$\Rightarrow \frac{1}{a} = \frac{2}{r} - \frac{2V^2}{\mu}$$

$$\Rightarrow a = \frac{1}{2}\left(\frac{1}{r}-\frac{V^2}{\mu}\right)^{-1}$$

Now the periodic time $T = \frac{2\pi}{\sqrt{\mu}}a^{3/2}$

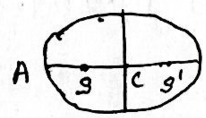
$$\therefore T = \frac{2\pi}{\sqrt{\mu}}\frac{1}{2\sqrt{2}}\left(\frac{1}{r}-\frac{V^2}{\mu}\right)^{-\frac{3}{2}}$$

$$= \frac{\pi}{\sqrt{2\mu}}\left(\frac{1}{r}-\frac{V^2}{\mu}\right)^{-\frac{3}{2}}.$$

(8) If ω be the angular velocity of a planet at the nearer end of the major axis, prove that its period is $\frac{2\pi}{\omega}\cdot\sqrt{\frac{(1+e)}{(1-e)^3}}$.

Soln
Let ω be the angular velocity of a planet at the nearer end

A of the major axis.



We have, $r^2\dot{\theta} = h$

Now, $r = SA = CA - CS = a - ae = a(1-e)$

Here $\dot{\theta} = \frac{d\theta}{dt} = \omega$

So $h = a^2(1-e)^2\omega$ — (I)

Again we have, $h = \sqrt{\mu l}$, where l is the semi latus rectum of the orbit

Then, $h^2 = \mu l = \mu a(1-e^2)$ (II), where e is the eccentricity of the elliptic orbit.

From (I) and (II),

$$\mu a(1-e^2) = \{a^2(1-e)^2\omega\}^2$$

$$\Rightarrow \mu a(1-e^2) = a^4(1-e)^4\omega^2$$

$$\Rightarrow a^3 = \frac{\mu(1-e)(1+e)}{(1-e)^4\omega^2} = \frac{\mu(1+e)}{\omega^2(1-e)^3}$$

$$\Rightarrow a = \left[\frac{\mu(1+e)}{\omega^2(1-e)^3}\right]^{\frac{1}{3}} \quad \text{— (III)}$$

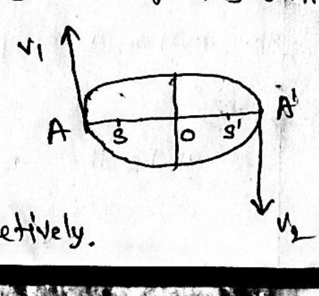
If T be the period of the orbit, then

$$T = \frac{2\pi}{\sqrt{\mu}} a^{3/2} = \frac{2\pi}{\sqrt{\mu}} \cdot \frac{\sqrt{\mu(1+e)}}{\omega\sqrt{(1-e)^3}}$$

$$= \cancel{\frac{2\pi}{\sqrt{\mu}}} \frac{2\pi}{\omega} \cdot \sqrt{\frac{1+e}{(1-e)^3}}. \quad \text{Ans}$$

(18) If v_1 and v_2 are the velocities of a planet when it is respectively nearest and farthest from the sun, prove that ~~from~~

$(1-e)v_1 = (1+e)v_2$, where e is the eccentricity of the planets orbit.



Soln. Let A, A' on the ellipse at the least and greatest distance from S, where the velocities of the planet are v_1 and v_2 respectively.

Now for the point A, we have from the relation $h = pv$.

$$h = AS\, v_1 = (OA - OS) v_1 = (a - ae) v_1 = a(1-e) v_1$$

Again for the point A', we have from the relation $h = pv$

$$h = A'S\, v_2 = (a + ae) v_2 = a(1+e) v_2 .$$

By Kepler's second law, $\frac{h}{2}$ is constant ie h h is constant

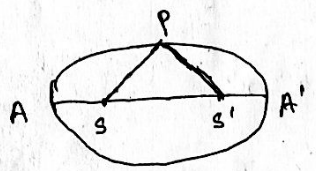
Therefore, $a(1-e) v_1 = a(1+e) v_2$

ie $(1-e) v_1 = (1+e) v_2$.

(23) If the velocity in a given elliptic orbit of major axis 2a is the same at a certain point p whether the orbit is ~~belong~~ being described in periodic time T about one focus S or in periodic time T' about the other focus S' prove that

$$SP = 2aT'/(T+T') , \quad S'P = 2aT/(T+T') .$$

Soln



Let AA' be the major axis of the ellipse and P be any point on the ellipse.

Let S and S' be the two focus of the ellipse.

Then $SP + S'P = 2a$ — (a)

Let $SP = r$, then $S'P = 2a - r$ — (i)

If v_1 be the velocity at the point P, with respect to the focus S then, $v_1^2 = \mu\left(\frac{2}{SP} - \frac{1}{a}\right)$ — (ii)

If T be the periodic time, then $T = \frac{2\pi}{\sqrt{\mu}} a^{3/2}$ — (iii)

Again if v_2 be the velocity at the point P with respect to the focus S', then $v_2^2 = \mu'\left(\frac{2}{S'P} - \frac{1}{a}\right)$ — (iv)

If T' be the periodic time, then $T' = \frac{2\pi}{\sqrt{\mu'}} a^{3/2}$ — (v)

By the given condition

$$V_1 = V_2$$

$$\Rightarrow \quad V_1^2 = V_2^2$$

$$\Rightarrow \quad \mu\left(\frac{2}{SP} - \frac{1}{a}\right) = \mu'\left(\frac{2}{S'P} - \frac{1}{a}\right)$$

$$\Rightarrow \quad \frac{4\pi^2 a^3}{T^2}\left(\frac{2}{SP} - \frac{1}{a}\right) = \frac{4\pi^2 a^3}{T'^2}\left(\frac{2}{S'P} - \frac{1}{a}\right)$$

using (iii) and (v)

$$\Rightarrow \quad T'^2\left(\frac{2}{r} - \frac{1}{a}\right) = T^2\left(\frac{2}{2a-r} - \frac{1}{a}\right)$$

using (i)

$$\Rightarrow \quad T'^2 \, \frac{2a-r}{ra} = T^2 \cdot \frac{[2a-(2a-r)]}{a(2a-r)}$$

$$\Rightarrow \quad T'^2(2a-r)^2 = T^2 r^2$$

$$\Rightarrow \quad \left(\frac{2a-r}{r}\right)^2 = \frac{T^2}{T'^2}$$

$$\Rightarrow \quad \frac{2a-r}{r} = \frac{T}{T'}$$

$$\Rightarrow \quad \frac{2a}{r} - 1 = \frac{T}{T'} \quad \text{--- (vi)}$$

~~Again, $S'P = 2a - r = 2a - \frac{2aT'}{T+T'}$~~ using (vi)

$$\Rightarrow \quad \frac{2a}{r} = \frac{T+T'}{T'}$$

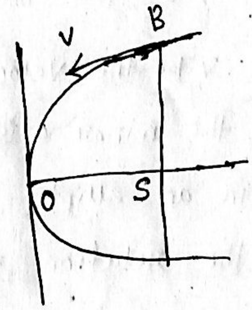
$$r = \frac{2aT'}{T+T'} = SP$$

Again, $S'P = 2a - r = 2a - \frac{2aT'}{T+T'} = \frac{2aT}{T+T'}$

So, $SP = \frac{2aT'}{T+T'}$ & $S'P = \frac{2aT}{T+T'}$

20 25. A particle is describing a parabola of latus rectum $4a$, under a force to the focus. When it is ~~and~~ at the end of the latus rectum its velocity is suddenly halved. Show that it now proceeds to describe an ellipse of major axis $\frac{8}{3}a$. What is the eccentricity of the ellipse?

Soln: Let, v be the velocity of the particle at B, at the end of the latus rectum, when the particle moving in a parabolic orbit.



Then from the relation $v^2 = \frac{2\mu}{r}$ we get

$$v^2 = \frac{2\mu}{2a} = \frac{\mu}{a} \quad \text{--- (i)}$$

Let the eqn of the new orbit be

$$v^2 = \frac{2\mu}{r} + A$$

Now when $r = 2a$, $v = \frac{V}{2}$

So, $$\frac{V^2}{4} = \frac{2\mu}{2a} + A$$

ie $$A = \frac{\mu}{4a} - \frac{\mu}{a} = -\frac{3\mu}{4a}$$

Then $$v^2 = \frac{2\mu}{r} - \frac{3\mu}{4a} = \mu\left(\frac{2}{r} - \frac{3}{4a}\right)$$

So the new orbit will be an ellipse, whose semi major axis is $\left(\frac{4a}{3}\right)$.

Let h', l', e' be the angular momentum, semi latus rectum and eccentricity of the new orbit.

Then $$h'^2 = \mu l' = \mu a'(1-e'^2) = \mu \frac{4a}{3}\cdot(1-e'^2) \quad \text{--- (ii)}$$

Again, $$h' = pv = \sqrt{2}a \cdot \frac{V}{2} = \frac{1}{\sqrt{2}} aV \quad \text{--- (iii)}$$

[For parabolic orbit, $p^2 = ax$
$p = \sqrt{ax}$
$= \sqrt{a \cdot 2a}$
$= \sqrt{2}a$]

From (ii) and (iii) we get

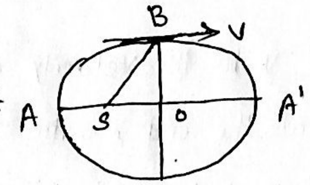
$$\frac{1}{2}a^2V^2 = \mu \cdot \frac{4a}{3}(1-e'^2)$$

$$\Rightarrow \frac{a^2}{2}\cdot\frac{\mu}{a} = \mu\frac{4a}{3}(1-e'^2) \quad \text{using (i)}$$

$$\Rightarrow e'^2 = \frac{5}{8} \quad \text{ie} \quad e' = \sqrt{\frac{5}{8}} \quad \text{Ans}$$

(27) A planet is describing elliptical orbit about the sun. When it is at an end of the minor axis suddenly its velocity is increased by half of its original velocity. Show that its orbit will be a hyperbola of eccentricity $\frac{1}{4}\sqrt{25-9e^2}$, where e is the eccentricity of the original orbit.

Soln Let v be the velocity at B at the end of the minor axis when the planet moving in an elliptic orbit.



So by the relation $v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$ we have,

$$v^2 = \mu\left(\frac{2}{a} - \frac{1}{a}\right) = \frac{\mu}{a}$$

[$\because$ $SB = a$ as
$SB^2 = a^2e^2 + b^2$
$= a^2e^2 + a^2(1-e^2)$
$= a^2$
$\therefore$ $SB = a$]

If v' be the new velocity at B.

Then $v' = \frac{3}{2}v$

$$\therefore \quad v'^2 = \frac{9}{4}v^2 = \frac{9\mu}{4a} \quad \text{--(i)}$$

Let the eqn of the new orbit be $v^2 = \frac{2\mu}{r} + A$ —(ii)

Now, when $r = a$, then $v = v'$. $\therefore$ $v'^2 = \frac{2\mu}{a} + A$

$$\text{i.e.} \quad A = \frac{2\mu}{a} - \frac{9\mu}{4a} = \frac{\mu}{4a} \quad \text{—(iii)}$$

There from from (ii) $v^2 = \frac{2\mu}{r} + \frac{\mu}{4a} = \mu\left(\frac{2}{r} + \frac{1}{4a}\right)$

which is a hyperbola, whose length of semi-transverse axis is $4a$.

Now, let h', l' & e' be the angular momentum, semi latus rectum and eccentricity of the new orbit.

Then $h'^2 = \mu l' = \mu a'(e'^2 - 1) = \mu . 4a(e'^2 - 1)$ --(iv)

Again $h' = pv = b.v'$

$$\Rightarrow \quad h'^2 = b^2v'^2 = a^2(1-e^2)\frac{9\mu}{4a} = \frac{9\mu(1-e^2).a}{4} \quad \text{(v)}$$

From (iv) and (v) we get

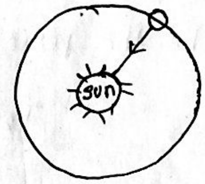
$$\mu 4a(e'^2 - 1) = \frac{9\mu(1-e^2).a}{4}$$

$$\Rightarrow \quad e'^2 = 1 + \frac{9}{16} - \frac{9e^2}{16} = \frac{25}{16} - \frac{9e^2}{16} = \frac{1}{16}(25 - 9e^2)$$

$$\Rightarrow \quad e' = \frac{1}{4}\sqrt{25 - 9e^2}$$

(30) If a planet were suddenly stopped in its orbit, supposed circular show that it will fall into the sun in a time which is $\left(\frac{\sqrt{2}}{8}\right)$ times the period of the planets revolution.

Soln when the planet is suddenly stopped in its orbit, it will be only attracted by the sun according to the law of inverse square and the motion of the planet takes place in a st. line towards to the sun.



The eqn of motion is

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^2} \quad -(1)$$

multiplying both side of (1) by $2.\frac{dx}{dt}$

$$2\frac{dx}{dt}\frac{d^2x}{dt^2} = -\frac{2\mu}{x^2}\frac{dx}{dt}$$

$$\Rightarrow \frac{d}{dt}\left\{\left(\frac{dx}{dt}\right)^2\right\} = -\frac{2\mu}{x^2}\left(\frac{dx}{dt}\right)$$

Integrating, $\left(\frac{dx}{dt}\right)^2 = \frac{2\mu}{x} + c$, where c is the arbitary constant

Since the planet is stopped in its orbit then

$\frac{dx}{dt} = 0$ when $x = a$, a being the radius of the circular orbit

so, $0 = \frac{2\mu}{a} + c$ i.e $c = -\frac{2\mu}{a}$

so, $\left(\frac{dx}{dt}\right)^2 = \frac{2\mu}{x} - \frac{2\mu}{a} = 2\mu\left(\frac{a-x}{ax}\right)$

$\Rightarrow \frac{dx}{dt} = -\sqrt{2\mu}.\sqrt{\frac{a-x}{ax}}$, Negative sign is taken as t increases x decreases.

$\Rightarrow \int_a^0 \frac{\sqrt{x}}{\sqrt{a-x}}\,dx = -\int_0^{T_1}\sqrt{\frac{2\mu}{a}}\,dt$. where T_1 is the time taken by planet to reach.

$\Rightarrow -\sqrt{\frac{2\mu}{a}}\,T_1 = \int_{\pi/2}^{0}\sqrt{\frac{a\sin^2\theta}{a\cos^2\theta}}\,2a\sin\theta\cos\theta\,d\theta$ Let $x = a\sin^2\theta$

$dx = 2a\sin\theta\cos\theta\,d\theta$

$x = a,\ \theta = \frac{\pi}{2}$

$\Rightarrow -\sqrt{\frac{2\mu}{a}}\,T_1 = a\int_{\pi/2}^{0}(1-\cos 2\theta)\,d\theta$ $x = 0,\ \theta = 0$

$\Rightarrow -\sqrt{\frac{2\mu}{a}}\,T_1 = a.\left[\theta - \frac{\sin 2\theta}{2}\right]_{\frac{\pi}{2}}^{0}$

$$-\sqrt{\frac{2\mu}{a}}\,T_1 = a\left[0 - 0 - \frac{\pi}{2} - 0\right]$$

$$\Rightarrow \quad T_1 = \frac{\sqrt{a}}{\sqrt{2}\sqrt{\mu}}\cdot\frac{\pi}{2}a = \frac{\sqrt{2}}{8}\cdot\frac{2\pi a^{3/2}}{\sqrt{\mu}}$$

If T_2 be the periodic time of the planet then

$$T_2 = \frac{2\pi}{\sqrt{\mu}}a^{3/2}$$

So, $\quad T_1 = \frac{\sqrt{2}}{8}\cdot T_2$.

(34) A planet is describing an ellipse about the sun as focus, show that its velocity away from the sun is greatest when the radius vector to the planet is at right angles to the major axis of the path and that it is then $\frac{2\pi ae}{T\sqrt{1-e^2}}$, where $2a$ is the major axis, e the eccentricity and T the periodic time.

Soln The polar eqn of the ellipse is

$$\frac{l}{r} = 1 + e\cos\theta, \quad e<1$$

$$\Rightarrow \quad lu = 1 + e\cos\theta \quad -(1) \quad \text{where } u = \frac{1}{r} \qquad \left[\frac{dr}{d\theta} = -\frac{1}{u^2}\frac{du}{d\theta} = -r^2\frac{du}{d\theta}\right]$$

we have, $\quad r^2\dot{\theta} = h$

$$\Rightarrow \quad r^2\frac{d\theta}{dt} = h$$

Again the radical velocity, $\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\cdot\frac{d\theta}{dt} = -r^2\frac{du}{d\theta}\frac{d\theta}{dt}$

$$= -h\frac{du}{d\theta}$$

Again from (1) $\quad \frac{du}{d\theta} = -\frac{e}{l}\sin\theta$

So, $\quad \frac{dr}{dt} = \frac{eh}{l}\sin\theta$

Now for maximum value of $\frac{dr}{dt}$, $\quad \frac{d}{d\theta}\left(\frac{dr}{dt}\right) = 0$

This gives $\quad \frac{eh}{l}\cos\theta = 0 \quad \therefore \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

Hence the 1st part is proved.

Now the maximum value of $\frac{dr}{dt}$ is

Now, $h = \sqrt{\mu l}$, $l = a(1-e^2)$, $T = \frac{2\pi}{\sqrt{\mu}}a^{3/2}$

$$\left(\frac{dr}{dt}\right)_{max} = \frac{eh}{l}\sin\frac{\pi}{2} = \frac{eh}{l} = \frac{e\sqrt{\mu\, a(1-e^2)}}{a(1-e^2)}$$

$$= \frac{1}{\sqrt{a(1-e^2)}}\cdot\frac{2\pi}{T}a^{3/2} = \frac{2\pi ea}{T\sqrt{1-e^2}} \quad \text{(Ans)}$$