Study Material
on
Theory of Motion

For PG Mathematics Students of Semester VIII (5 Yr. Int. M. Sc.)
& Semester II (2 Yr. M. Sc.)
Course Code-MA412 & MT412, Year 2019 - 2020

1 Path line

A path line is the curve or path followed by a particle during motion or flow.

2 Stream line

A stream line is the curve whose tangent at any point is in the direction of the velocity at that point.

2.1 Differential equation of stream line

At every point on a streamline the tangent is in the direction of the velocity $\vec{v} = (v_1, v_2, v_3)$. Hence for the differential tangent vector $\vec{dx} = (dx_1, dx_2, dx_3)$ along the streamline, $\vec{v} \times \vec{dx} = \vec{0}$ and accordingly the differential equations of the streamlines become $\frac{dx_1}{v_1} = \frac{dx_2}{v_2} = \frac{dx_3}{v_3}$.

3 Steady motion

The motion of a continuum is termed steady motion if the velocity field is independent of time so that $\frac{\partial v_i}{\partial t} = 0$. For steady motion, stream lines and path lines coincide.

4 Rate of deformation tensor/rate of strain/ stretching

The symmetric tensor

$$d_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

is called rate of deformation tensor.

Since d_{ij} is a symmetric, second-order tensor, the concepts of principal axes, principal values, invariants may be associated with it.

5 Vorticity or spin tensor

The skew symmetric tensor

$$w_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)$$

is called vorticity or spin tensor.

6 Vorticity vector

The vorticity vector $\vec{w}$ is the curl of the velocity field, i.e., $\vec{w} = \vec{\nabla} \times \vec{v} = curl \vec{v}$.

7 Rate of rotation tensor

The vector $\vec{\Omega}$ defined as one-half the vorticity vector, i.e.,

$$\vec{\Omega} = \frac{1}{2}\vec{w}$$

is called the rate of rotation vector.

8 Irrotational motion

The motion of a continuum is termed as irrotational if $\vec{w} = \vec{\nabla} \times \vec{v} = curl \vec{v} = \vec{0}$ at every point of the continuum.

9 Velocity potential

If there exists a scalar function ϕ such that $\vec{v} = -\vec{\nabla}\phi$, i.e., $v_i = -\frac{\partial \phi}{\partial x_i}$, then ϕ is called the velocity potential.

10 Relation between irrotational motion and velocity potential

$\vec{\nabla} \times \vec{v} = \vec{0}$ is necessary and sufficient condition for a velocity potential ϕ to exist. So the velocity vector for irrotational motion may be expressed by $\vec{v} = -\vec{\nabla}\phi$, or $v_i = -\frac{\partial \phi}{\partial x_i}$.

11 Vortex line

A vortex line is one whose tangent at every point in a moving continuum is in the direction of the vorticity vector $\vec{w}$.

11.1 Differential equation of vortex lines

Let $\vec{dx} = (dx_1, dx_2, dx_3)$ be a differential distance vector in the direction of $\vec{w} = (w_1, w_2, w_3)$. Then $\vec{w} \times \vec{dx} = \vec{0}$, or

$$(w_2 dx_3 - w_3 dx_2)\vec{e}_1 + (w_3 dx_1 - w_1 dx_3)\vec{e}_2 + (w_1 dx_2 - w_2 dx_1)\vec{e}_3 = \vec{0},$$

from which we get

$$\frac{dx_1}{w_1} = \frac{dx_2}{w_2} = \frac{dx_3}{w_3},$$

or

$$\frac{dx_1}{\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}} = \frac{dx_2}{\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}} = \frac{dx_3}{\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}}$$

which are differential equations for vortex lines.

Problem 1

A continuum motion is expressed by

$$x_1 = X_1, \tag{1}$$

$$x_2 = \frac{e^t(X_2 + X_3)}{2} + \frac{e^{-t}(X_2 - X_3)}{2}, \tag{2}$$

$$x_3 = \frac{e^t(X_2 + X_3)}{2} - \frac{e^{-t}(X_2 - X_3)}{2}. \tag{3}$$

Determine the velocity components in both their material and spatial forms.

Solution: From the second and third equations,

$$X_2 + X_3 = e^{-t}(x_2 + x_3), \tag{4}$$

$$X_2 - X_3 = e^{t}(x_2 - x_3). \tag{5}$$

Solving these simultaneously the inverse equations become

$$X_1 = x_1, \tag{6}$$

$$X_2 = \frac{e^{-t}(x_2 + x_3)}{2} + \frac{e^t(x_2 - x_3)}{2}, \tag{7}$$

$$X_3 = \frac{e^{-t}(x_2 + x_3)}{2} - \frac{e^t(x_2 - x_3)}{2}, \tag{8}$$

Accordingly, the displacement components $u_i = x_i - X_i$ may be written in either the Lagrangian form

$$u_1 = 0,$$

$$u_2 = \frac{e^t(X_2+X_3)}{2} + \frac{e^{-t}(X_2-X_3)}{2} - X_2,$$
$$u_3 = \frac{e^t(X_2+X_3)}{2} - \frac{e^{-t}(X_2-X_3)}{2} - X_3,$$

or in the Eulerian form

$$u_1 = 0,$$
$$u_2 = x_2 - \frac{e^{-t}(x_2+x_3)}{2} - \frac{e^t(x_2-x_3)}{2},$$
$$u_3 = x_3 - \frac{e^{-t}(x_2+x_3)}{2} + \frac{e^t(x_2-x_3)}{2}.$$

The velocity components in Lagrangian form are

$$v_1 = \frac{\partial u_1}{\partial t} = 0,$$
$$v_2 = \frac{\partial u_2}{\partial t} = \frac{e^t(X_2+X_3)}{2} - \frac{e^{-t}(X_2-X_3)}{2}$$
$$v_3 = \frac{\partial u_3}{\partial t} = \frac{e^t(X_2+X_3)}{2} + \frac{e^{-t}(X_2-X_3)}{2}.$$

Using the relationships (4) and (5) these components reduce to

$$v_1 = 0,\ v_2 = x_3,\ v_3 = x_2,$$

which are velocity components in spatial form.

These can also be obtained as follows

$$v_1 = \frac{du_1}{dt} = \frac{\partial u_1}{\partial t} + v_1\frac{\partial u_1}{\partial x_1} + v_2\frac{\partial u_1}{\partial x_2} + v_3\frac{\partial u_1}{\partial x_3} = 0,$$

$$v_2 = \frac{du_2}{dt} = \frac{\partial u_2}{\partial t} + v_1\frac{\partial u_2}{\partial x_1} + v_2\frac{\partial u_2}{\partial x_2} + v_3\frac{\partial u_2}{\partial x_3} = \frac{e^{-t}(x_2+x_3)}{2} - \frac{e^t(x_2-x_3)}{2} + v_2\frac{2-e^{-t}-e^t}{2} + v_3\frac{-e^{-t}+e^t}{2},$$

$$v_3 = \frac{du_3}{dt} = \frac{\partial u_3}{\partial t} + v_1\frac{\partial u_3}{\partial x_1} + v_2\frac{\partial u_3}{\partial x_2} + v_3\frac{\partial u_3}{\partial x_3} = \frac{e^{-t}(x_2+x_3)}{2} + \frac{e^t(x_2-x_3)}{2} + v_2\frac{-e^{-t}+e^t}{2} + v_3\frac{2-e^{-t}-e^t}{2}.$$

Solving these equations simultaneously for v_2 and v_3, the result is as before $v_1 = 0, v_2 = x_3, v_3 = x_2$.

Problem 2

A velocity field is described by $v_1 = \frac{x_1}{(t+1)}, v_2 = \frac{2x_2}{(t+1)}, v_3 = \frac{3x_3}{(t+1)}$. Determine the acceleration components for this motion.

Solution: $f_1 = \frac{dv_1}{dt} = \frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} + v_3 \frac{\partial v_1}{\partial x_3} = -\frac{x_1}{(1+t)^2} + \frac{x_1}{(t+1)^2} = 0$

$f_2 = \frac{dv_2}{dt} = \frac{\partial v_2}{\partial t} + v_1 \frac{\partial v_2}{\partial x_1} + v_2 \frac{\partial v_2}{\partial x_2} + v_3 \frac{\partial v_2}{\partial x_3} = -\frac{2x_2}{(1+t)^2} + \frac{4x_2}{(t+1)} \frac{1}{(t+1)^2} = \frac{2x_2}{(t+1)^2}$

$f_3 = \frac{dv_3}{dt} = \frac{\partial v_3}{\partial t} + v_1 \frac{\partial v_3}{\partial x_1} + v_2 \frac{\partial v_3}{\partial x_2} + v_3 \frac{\partial v_3}{\partial x_3} = -\frac{3x_3}{(1+t)^2} + \frac{9x_3}{(t+1)} \frac{1}{(t+1)^2} = \frac{6x_3}{(t+1)^2}$

Problem 3

A velocity field is described by $v_1 = \frac{x_1}{(t+1)}, v_2 = \frac{2x_2}{(t+1)}, v_3 = \frac{3x_3}{(t+1)}$. Determine the acceleration components in Lagrangian form for the motion.

Solution: From first relation we get $\frac{dx_1}{dt} = \frac{x_1}{(t+1)} \Rightarrow \frac{dx_1}{x_1} = \frac{dt}{1+t}$ which upon integration gives $\ln x_1 = \ln(1+t) + \ln C$, where C is a constant of integration. Since $x_1 = X_1$ when $t = 0$, $C = X_1$ and so $x_1 = X_1(1+t)$. Similar integrations will give us $x_2 = X_2(1+t)^2$ and $x_3 = X_3(1+t)^3$. So, $v_1 = \frac{\partial u_1}{\partial t} = X_1,\ v_2 = \frac{\partial u_2}{\partial t} = 2X_2(1+t), v_3 = \frac{\partial u_3}{\partial t} = 3X_3(1+t)^2$ and $f_1 = \frac{\partial v_1}{\partial t} = 0,\ f_2 = \frac{\partial v_2}{\partial t} = 2X_2,\ f_3 = \frac{\partial v_3}{\partial t} = 6X_3(1+t)$.

Problem 4

The motion of a continuum is given by $x_1 = X_1 + X_2 t + X_3 t^2$, $x_2 = X_2 + X_3 t + X_1 t^2$, $x_3 = X_3 + X_1 t + X_2 t^2$. Find at time t, the velocity and acceleration of the particle (a) which occupies the point (1,1,1) at t, (b) which was at (1,1,1) at $t = 0$.

Problem 5

A velocity field is specified by the vector $\vec{v} = x_1^2 t \vec{e}_1 + x_2 t^2 \vec{e}_2 + x_1 x_3 t \vec{e}_3$. Determine the velocity and acceleration of the particle at P(l,3,2) when t = 1.

Ans: $\vec{v} = \vec{e}_1 + 3\vec{e}_2 + 2\vec{e}_3,\ \vec{f} = 3\vec{e}_1 + 9\vec{e}_2 + 6\vec{e}_3$.

Problem 6

The magnetic field strength of an electromagnetic continuum is given by $\lambda = \frac{e^{-At}}{r}$ where $r^2 = x_1^2 + x_2^2 + x_3^2$ and A is a constant. If the velocity field of the continuum is given by $v_1 = Bx_1x_3t,\ v_2 = Bx_2^2t^2,\ v_3 = Bx_3x_2$, determine the rate of change of magnetic intensity for the particle at $P(2,-1,2)$ when $t=1$.

Solution:

$$\dot{\lambda} = \frac{d\lambda}{dt} = -A\frac{e^{-At}}{r} - \frac{e^{-At}}{r^2}\frac{dr}{dt}.$$

Now

$$\begin{aligned} \tfrac{dr}{dt} &= \frac{\partial r}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial r}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial r}{\partial x_3}\frac{dx_3}{dt} \\ &= v_1\frac{\partial r}{\partial x_1} + v_2\frac{\partial r}{\partial x_2} + v_3\frac{\partial r}{\partial x_3} \\ &= Bx_1x_3t\frac{\partial r}{\partial x_1} + Bx_2^2t^2\frac{\partial r}{\partial x_2} + Bx_3x_2\frac{\partial r}{\partial x_3} \\ &= \frac{Bx_1^2x_3t + Bx_2^3t^2 + Bx_3^2x_2}{r} \end{aligned}$$

since $\frac{\partial r}{\partial x_i} = \frac{x_i}{r}, i = 1,2,3$. Therefore,

$$\frac{d\lambda}{dt} = -A\frac{e^{-At}}{r} - \frac{e^{-At}(Bx_1^2x_3t + Bx_2^3t^2 + Bx_3^2x_2)}{r^3}.$$

Thus for $P(2,-1,2)$ at $t=1$, $\frac{d\lambda}{dt} = -A\frac{e^{-A}}{3} - \frac{e^{-A}3B}{3^3} = -\frac{e^{-A}(3A+B)}{9}$.

Problem 7

For the velocity field $v_1 = 3x_1^2x_2$, $v_2 = 2x_2^2x_3$, $v_3 = x_1x_2x_3^2$, determine the rate of extension at P(1,1,1) in the direction $(\frac{3}{5}, 0, -\frac{4}{5})$.

[Hint: Calculate d_{ij} at $P(1,1,1)$ and then use the formula $d_{(n)} = d_{ij}n_in_j$, where $(n_1,n_2,n_3) = (\frac{3}{5}, 0, -\frac{4}{5})$

$$(d_{ij}) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) = \begin{pmatrix} 6x_1x_2 & \frac{3x_1^2}{2} & \frac{x_2x_3^2}{2} \\ \frac{3x_1^2}{2} & 4x_2x_3 & \frac{2x_2^2+x_1x_3^2}{2} \\ \frac{x_2x_3^2}{2} & \frac{2x_2^2+x_1x_3^2}{2} & 2x_1x_2x_3 \end{pmatrix},$$

At (1,1,1)

$$(d_{ij}) = \begin{pmatrix} 6 & \frac{3}{2} & \frac{1}{2} \\ \frac{3}{2} & 4 & \frac{3}{2} \\ \frac{1}{2} & \frac{3}{2} & 2 \end{pmatrix}$$

Ans:$\frac{74}{25}$]

Problem 8

For velocity field $v_1 = 3x_1^2 x_2$, $v_2 = 2x_2^2 x_3$,$v_3 = x_1 x_2 x_3^2$, determine the rate of shear at the point (1,1,1) between orthogonal direction $(\frac{3}{5}, 0, \frac{-4}{5})$ and $(\frac{4}{5}, 0, \frac{3}{5})$.

[Hint: Calculate d_{ij} at $P(1,1,1)$ and then use the formula $S_{(nm)} = 2d_{ij}n_i m_j$, where $(n_1, n_2, n_3) = (\frac{3}{5}, 0, \frac{-4}{5})$, $(m_1, m_2, m_3) = (\frac{4}{5}, 0, \frac{3}{5})$ and

$$(d_{ij}) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) = \begin{pmatrix} 6x_1x_2 & \frac{3x_1^2}{2} & \frac{x_2x_3^2}{2} \\ \frac{3x_1^2}{2} & 4x_2x_3 & \frac{2x_2^2+x_1x_3^2}{2} \\ \frac{x_2x_3^2}{2} & \frac{2x_2^2+x_1x_3^2}{2} & 2x_1x_2x_3 \end{pmatrix}.$$

At (1,1,1)

$$(d_{ij}) = \begin{pmatrix} 6 & \frac{3}{2} & \frac{1}{2} \\ \frac{3}{2} & 4 & \frac{3}{2} \\ \frac{1}{2} & \frac{3}{2} & 2 \end{pmatrix}.$$

Ans: $\frac{89}{25}$]

Problem 9

For the velocity field $v_1 = x_1^2 x_2 + x_2^3$, $v_2 = -(x_1^3 + x_1 x_2^2)$, $v_3 = 0$, determine the rate of extension at P(1,2,3) in the direction $(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3})$.

[Hint: Calculate d_{ij} at $P(1,2,3)$ and then use the formula $d_{(n)} = d_{ij}n_i n_j$, where $(n_1, n_2, n_3) = (\frac{1}{3}, -\frac{2}{3}, \frac{2}{3})$,

$$(d_{ij}) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) = \begin{pmatrix} 2x_1x_2 & -x_1^2 + x_2^2 & 0 \\ -x_1^2 + x_2^2 & -2x_1x_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

At (1,2,3)

$$(d_{ij}) = \begin{pmatrix} 4 & 3 & 0 \\ 3 & -4 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Ans: $-\frac{24}{9}$]

Problem 10

For a steady field $v_1 = x_1^2 x_2 + x_2^3$, $v_2 = -(x_1^3 + x_1 x_2^2)$, $v_3 = 0$, determine the principal strain-rates at $P(x_1, x_2, x_3)$. [Hint:

$$(d_{ij}) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) = \begin{pmatrix} 2x_1x_2 & -x_1^2 + x_2^2 & 0 \\ -x_1^2 + x_2^2 & -2x_1x_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Principal strain rates are the roots of the equation

$$\begin{vmatrix} 2x_1x_2 - d & -x_1^2 + x_2^2 & 0 \\ -x_1^2 + x_2^2 & -2x_1x_2 - d & 0 \\ 0 & 0 & -d \end{vmatrix} = 0.$$

Ans: $d_1 = x_1^2 + x_2^2, d_2 = 0, d_3 = -(x_1^2 + x_2^2)$]

Problem 11

For the velocity field described by $v_1 = \frac{x_1}{(t+1)}, v_2 = \frac{2x_2}{(t+1)}, v_3 = \frac{3x_3}{(t+1)}$, determine the streamlines and path lines of the flow and show that they coincide.

Solution: For the given flow differential equatins of the streamlines are are $\frac{dx_1}{v_1} = \frac{dx_2}{v_2} = \frac{dx_3}{v_3}$, i.e., $\frac{dx_1}{x_1} = \frac{dx_2}{2x_2} = \frac{dx_3}{3x_3}$. Integrating and using the conditions $x_i = X_i$ when $t = 0$, the equations of the streamlines are $(\frac{x_1}{X_1})^2 = \frac{x_2}{X_2}$, $(\frac{x_1}{X_1})^3 = \frac{x_3}{X_3}$, $(\frac{x_2}{X_2})^3 = (\frac{x_3}{X_3})^2$.

Integration of the velocity expressions $\frac{dx_i}{dt} = v_i$ yields the displacement equations $x_1 = X_1(1+t)$, $x_2 = X_2(1+t)^2$ and $x_3 = X_3(1+t)^3$. Eliminating t from these equations gives the path lines which are identical with the streamlines presented above.

Problem 12

For the velocity field $v_1 = \frac{3x_1^2 - r^2}{r^5}$, $v_2 = \frac{3x_1x_2}{r^5}$, $v_3 = \frac{3x_1x_3}{r^5}$, where $r^2 = x_1^2 + x_2^2 + x_3^2$, show that motion is irrotational. Find the velocity potential. Show that stream lines are given by $(x_1^2 + x_2^2 + x_3^2)^3 = c_1(x_2^2 + x_3^2)^2$ and $x_2 = c_2 x_3$.

Problem 13

Find the stream line and path line of a continuum particle for the velocity field $v_1 = \frac{x_1}{1+t}, v_2 = x_2, v_3 = 0$.

Problem 14

If $\phi = (x_1 - t)(x_2 - t)$ be the velocity potential of the two dimensional irrotational motion of a continuum, then show that stream lines at time t are $(x_1 - t)^2 - (x_2 - t)^2 =$ constant, and that paths of particles have the equations $\log|x_1 - x_2| = \frac{1}{2}[(x_1 + x_2) - a(x_1 - x_2)^{-1}] + b$, where a, b are constants.

Problem 15

For the velocity field given by $v_1 = kx_3$, $v_2 = kx_3$, $v_3 = k(x_1 + x_2)$, show that the motion is irrotational. Find the velocity potential and show that stream lines are given by $x_1 - x_2 = c_1$ and $(x_1 + x_2)^2 - 2x_3^2 = c_2$.

Problem 16

A steady velocity field is given by $v_1 = 2x_3, v_2 = 2x_3, v3 = 0$. Determine the principal directions and principal values (rates of extension) of the rate of deformation tensor for this motion.
[Hint:

$$(d_{ij}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Principal strain rates are the roots of the equation

$$\begin{vmatrix} -d & 0 & 1 \\ 0 & -d & 1 \\ 1 & 1 & d \end{vmatrix} = 0.$$

Ans: $d_1 = \sqrt{2}$, $d_2 = 0$, $d_3 = -\sqrt{2}$, $(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}})$, $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0)$, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}})$]

Problem 17

Show that for the velocity field $\vec{v} = (Ax_3 - Bx_2)\vec{e}_1 + (Bx_1 - Cx_3)\vec{e}_2 + (Cx_2 - Ax_1)\vec{e}_3$ the vortex lines are straight lines and determine their equations. Also show that above velocity field represents a rigid body rotation.

Solution: We know that, $\vec{q} = \vec{\nabla} \times \vec{v} = 2(C\vec{e}_1 + A\vec{e}_2 + B\vec{e}_3)$. So the differential equations for the vortex lines are $Adx_3 = Bdx_2, Bdx_1 = Cdx_3, Cdx_2 = Adx_1$. Integrating these we get equations of the vortex lines as follows $x_3 = \frac{B}{A}x_2 + K_1, x_1 = \frac{C}{B}x_3 + K_2, x_2 = \frac{A}{C}x_1 + K_3$ where the K_i are constants of integration.

Since $(d_{ij}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, above velocity field represents a rigid body rotation.

Acknowledgement

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References

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2. George E Mase, Continuum Mechanics, Schaum's Outline Series, McGraw Hill Book Company, 1970.