COMPACTNESS OF METRIC SPACES

FOR THE MATHEMATICS UG STUDENTS OF SEMESTER VI
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1. Motivation

Let us first observe some remarkable properties of the closed interval $[a, b]$ of real line.

Theorem 1.1. (Bolzano–Weierstrass theorem) *Every bounded sequence of real numbers has a convergent subsequence.*

Another version of this theorem is

Theorem 1.2. (Bolzano–Weierstrass theorem (rephrased)) *Let X be any closed bounded subset of the real line. Then any sequence (x_n) of points in X has a subsequence converging to a point of X.*

Note that above property of X does not hold if X fails to be closed or fails to be bounded (why?).
Now look at the another important property of $[a, b]$.

Theorem 1.3. (Heine–Borel theorem) *Every covering of a closed interval $[a, b]$ — or more generally of a closed bounded set $X \subset \mathbb{R}$ — by a collection of open sets has a finite subcovering.*

These theorems are not only interesting but they are extremely useful for the purpose of applications. We also know that every real valued continuous function on a closed and bounded interval $[a, b]$ is bounded and attains its supremum and infemum on $[a, b]$.Not only that continuity of the function is strengthened to uniform continuity. Naturally, the question is that for what property of this closed and bounded intervals posses? Now our goal is to find out the generalization of the above results in arbitrary metric spaces.

2. Basic definitions and Results

Let (X, d) be a metric space.

Definition 2.1. (Cover(open Cover) or Covering(open covering)) A cover(open cover) or covering(open covering) of X is a collection of sets (open sets) whose union is X. Let X be a nonempty set and $A \subset X$. A collection U of subsets of X is called a cover or covering of the set A if $A \subset \bigcup U$.

Definition 2.2. (subcover) A subcollection $\mathbf{U_0}$ of $\mathbf{U}$ is called a **subcover** or a **subcovering** of $\mathbf{U}$ for A if $\mathbf{U_0}$ is also a cover of A. If the subcover $\mathbf{U_0}$ consits finite number of elements then it is called finite subcover.

Open covers of subsets of the Euclidean spaces $\mathbb{R}^n$ can always be reduced to countable ones, as the next classical result of E.Lindelöf shows.

Lemma 2.3. (Lindelöf) *Every open cover of a subset of* $\mathbb{R}^n$ *can be reduced to an at-most countable subcover.*

Proof. We call a point $a = (a_1, a_2, ..., a_n) \in \mathbb{R}^n$ rational, if each component a_i is a rational number. Let $A \subset \mathbb{R}^n$ and $\{U_i\}_{i\in J}$ is an infinite open cover of A. Now, for each $x \in A$, there exists $i_x \in J$ such that $x \in U_{i_x}$. Now we can choose a rational point $a_x \in \mathbb{R}^n$ and a positive rational numberr_x such that $x \in B(a_x, r_x) \subset U_{i_x}$. Then the collection $\{B(a_x, r_x) : x \in A\}$ is an at-most countable open cover of A. Since each $B(a_x, r_x)$ is a subset of some U_i, therefore there exists an at-most countable subcover of $\{U_i\}_{i\in J}$ for A. □

Now we are ready to define compactness.

Definition 2.4. (Compact metric space) Let (X, d) be a metric space. A subset A of X is said to be **compact** if every open cover of A can be reduced to a finite subcover. If X is itself a compact set, then (X,d) is referred to as a compact metric space.

We can rephrase compactness in terms of closed sets by making the following observation: If U is an open covering of X, then the collection F of complements of sets in U is a collection of closed sets whose intersection is empty (why?); and conversely, if F is a collection of closed sets whose intersection is empty, then the collection U of complements of sets in F is an open covering. Thus, a space X is compact if and only if every collection of closed sets with an empty intersection has a finite subcollection whose intersection is also empty. Or, passing to the contrapositive, we can put it another way by making the following definition:

Definition 2.5. (Finite Intersection Property) A collection F of sets is said to have the finite intersection property (f.i.p., for short) if every finite subcollection of F has a nonempty intersection.

From the above discussion we have shown that

Theorem 2.6. *A metric space X is compact if and only if every collection* F *of closed sets in X with the finite intersection property has a nonempty intersection.*

Note. The above theorem shows that in a compact metric space (X, d) if $F_1 \supseteq F_2 \supseteq ....$ is a descending sequence of nonempty closed sets, then $\bigcap_{n=1}^{\infty} F_n \neq \phi$ and therefore by cantor's intersection theorem, (X, d) is complete.

It is therefore natural to ask, whether complete metric space is compact or not. In general the answer is no. As you know that $\mathbb{R}$ is complete but not compact. After some time we will return to the topic again.
Before we discuss further we will give some example of compact metric spaces.

Example. Every finite subset of any metric space is always compact.

Note that no infinite subset of a discrete metric space is compact, as single tone sets are open.

Example. The real line $\mathbb{R}$ is not compact, as the open cover $\{(-n, n) : n \in \mathbb{N}\}$ of $\mathbb{R}$ have no finite subcover.

Example. Let (a_n) be a sequence in a metric space X, converging to a point$a$ in X. Let A denote the range of the sequence and $B = A \bigcup \{a\}$, then B is a compact subset of X. In fact, let F be any open cover of B, then there exists $U \in F$ such that $a \in U$. Since U is an open set and a is the limit of the sequence, hence all but finitely elements of B is in U and the rest of the elements of B must be in finite number of elements of F and they together with U form a finite subcover of B.

Now we will show another interesting feature of compact metric space.

Theorem 2.7. *Every closed subset of a compact metric space is compact.*

Proof. Let A be a closed subset of a compact metric space (X, d) and $\mathbf{F} = \{F_i : i \in J\}$ be an open cover of A. Clearly $\{F_i : i \in J\} \cup (X - A)$ is an open cover of X. By the compactness of X, there exists $F_1, F_2, ...F_n \in \mathbf{F}$ for some $n \in \mathbb{N}$ such that $X = \bigcup_{i=1}^{n} F_i \cup (X - A)$. Clearly $\{F_1, F_2, ...F_n\}$ covers A and hence A is compact. □

Now we will characterize the compact subsets of a metric space.

Theorem 2.8. *A compact subset of a metric space (x, d) is closed and bounded.*

Proof. Let Y be a compact subset of a metric space (X, d). To show Y to be closed we only have to show that any point of Y^c is not a limit point of Y.
So, let $x_0 \in Y^d$, and for each $y \in Y, d(y, x_0) = r_y$(say). Then $\mathbf{U} = \{B(y, \frac{r_y}{2}) : y \in Y\}$ is an open cover of Y. As Y is compact $\mathbf{U}$ has an finite subcover and hence there exists $\{y_1, y_2, ..., y_n\} \subseteq Y\}$ such that $Y \subseteq \bigcup_{i=1}^{n} B(y_i, \frac{r_{y_i}}{2})$. Let $r = min(r_{y_1}, r_{y_2}, ..., r_{y_n}) > 0$. Now we claim that $B(x_0, \frac{r}{2}) \cap Y = \phi$. If not, let $x \in B(x_0, \frac{r}{2}) \cap Y$. As, $Y \subseteq \bigcup_{i=1}^{n} B(y_i, \frac{r_{y_i}}{2})$, therefore say $x \in B(y_i, \frac{r_{y_i}}{2}) \Rightarrow d(x, y_i) < \frac{r_{y_i}}{2} \Rightarrow d(x, x_0) \geq \frac{r}{2}$. Hence $x \notin B(x_0, \frac{r}{2})$, a contradiction, which proves that $x_0 \notin Y^d$. Thus Y is closed.
Now we will show that Y is bounded. Since $Y \subseteq \bigcup_{x \in Y} B(x, 1)$, there exists a finite number ofpoints $x_1, x_2, ..., x_n$ of Y (as Y is compact) such that $Y \subseteq \bigcup_{i=1}^{n} B(x_i, 1). Let$M = max {d(x$_i, x_j)$: $i, j = 1, 2, .., n. If$x,y $\in Y$,

then choose i and j such that $x \in B(x_i, 1)$ and $y \in B(x_j, 1)$.Therefore, $d(x, y) \leq d(x, x_i) + d(x_i, x_j) + d(x_j, y) < M + 2 < \infty$, so Y is bounded. □

From the above two theorems we can conclude that, in a compact metric space (X, d), a subset A of X is compact iff it is closed in X.
Note: Converse of the above theorem need not be true. If we consider a discrete metric space$X, d)$, then any infinite subset A of X is closed and bounded but not compact(why?).
On the other hand, if a metric space (X, d) contains a noncompact closed subset A, then all the supersets of A are noncompact. As for an example $\mathbb{N}$ is a noncompact closed subsets of $\mathbb{R}$ and hence any superset of $\mathbb{N}$ if noncompact and in particular $\mathbb{Q}$, $\mathbb{R}$ are not compact.
To move further, we have to familiar with some other important notions.

Definition 2.9. (Sequentially compact) A metric space X is said to be sequentially compact if every sequence (x_n) of points in X has a convergent subsequence.

Note that, this abstracts the Bolzano–Weierstrass property; indeed, the Bolzano–Weierstrass theorem states that closed bounded subsets of the real line are sequentially compact.

Definition 2.10. (Total boundedness) A subset E of a metric space X is said to be toatally bounded (or precompact) if for every $\epsilon > 0$, there are $x_1, x_2, ..., x_n$ in X such that $E \subset B(x_1, \epsilon) \bigcup B(x_2, \epsilon) \bigcup ... \bigcup B(x_n, \epsilon)$.

Remark 2.11. If E is totally bounded, then we can find such $x_1, x_2, ..., x_n$ in E itself.

Remark 2.12. Every totally bounded set is bounded (why?), but a bounded set need not be totally bounded. For example, let $X = \mathbb{R}$ with the metric d given by $d(x, y) = min\{1, |x - y|\}$ for $x, y \in \mathbb{R}$. Then X is clearly bounded but not totally bounded since for $0 < \epsilon < 1$ and for any $x_1, x_2, ..., x_n \in \mathbb{R}$, $1 + x_1 + x_2 + ... + x_n \notin B_d(x_1, \epsilon) \bigcup B_d(x_2, \epsilon) \bigcup ... \bigcup B_d(x_n, \epsilon)$.

Remark 2.13. Every subset of a totally bounded set is totally bounded and so is its closure.(why?)

The next result characterizes the compact sets in metric spaces and gives an indication of the usefulness of the compact sets.

Theorem 2.14. *For a subset A of a metric space (X, d) the following statements are equivalent:*

(1) *A is a Compact set.*
(2) *Every infinite subset of A has an accumulation point in A.*
(3) *every sequence in A has a subseqence which converges to a point of A*
(4) *A is complete and totally bounded.*

Proof. (**1**) $\Longrightarrow$ (**2**) Let S be an infinite subset of a compact set A. Let if possible S has no accumulation point in A. Then, for every $x \in A$ there exists some $r_x > 0$ such that $B(x, r_x) \cap (S \setminus \{x\}) = \phi$. Therefore, $B(x, r_x) \cap S \subseteq \{x\}$. Clearly, $A \subseteq \bigcup_{x \in A} B(x, r_x)$ holds, and, in view of compactness of A, there exist $x_1, x_2, ..., x_n \in A$ such that $A \subseteq \bigcup_{i=1}^{n} B(x_i, r_{x_i})$. But then $S = A \cap S \subseteq \bigcup_{i=1}^{n} [B(x_i, r_{x_i}) \cap S] \subseteq \{x_1, x_2, ..., x_n\}$, shows that S must be a finite set, a contradiction. hence the proof.
(**2**) $\Longrightarrow$ (**3**) Let (x_n) be a sequence of A. If the range of the sequence is finite, then there is nothing to show- as it must have a constant subsequence. Now, if range of (x_n) is infinite, then there exist a subsequence (y_n) of (x_n) such that $y_n \neq y_m$ for $n \neq m$. Now the set $\{y_1, y_2, ...\}$ is an infinite subset of A, and by our hypothesis it has an accumulation point in A, say x. We can assume that $y_n \neq x$ for each n; if $y - k = n$, then replace (y_n) by (y_{n+k}). Choose m_1 with $d(y_{m_1}, x) < 1$ (such m_1 exists as x is an accumulation point). Now, inductively, if $m_1 < m_2 < ... < m_n$ have been selected, choose m_{n+1} such that $d(y_{m_{n+1}}, x) < min\{\frac{1}{n+1}, d(y_1, x), d(y_2, x), ..., d(y_m, x)\}$. Clearly, $m_{n+1} > m_n$ must hold. This shows that (y_{m_n}) is a subsequence of (y_n), and hence, a subsequence of (x_n). In view of $d(y_m, x) < \frac{1}{n}$, it follows that $lim y_{m_n} = x$, as required.
(**3**) $\Longrightarrow$ (**4**) Suppose that every sequence of A has a convergent subsequence. Since a A Cauchy sequence having a convergent subsequence is itself convergent, hence A is complete.
To show that A is totally bounded, let if possible A is not totally bounded. Then there is some $\epsilon > 0$ such that A can not by cover by finite number of open balls of radius ϵ. Let $x_1 \in A$. Then we can find x_2 in A such that $x_2 \notin B(x_1, \epsilon)$. Again, we have $x_3 \in A$ so that $x_3 \notin B(x_1, \epsilon) \cup B(x_2, \epsilon)$. In this way after nth choice we have $x_{n+1) \in A}$ such that $x_{n+1} \notin B(x_1, \epsilon) \cup B(x_2, \epsilon) \cup ...(x_n, \epsilon)$, and this is true for any $n \in \mathbb{N}$. Then $d(x_n, x_m) \geq \epsilon, \forall n, m = 1, 2, ...$, so that (x_n) cannot have a convergent subsequence, cotradictory to our hypothesis. Hence A is totally bounded.
(**3**) $\Longrightarrow$ (**4**) Let if possible A is not compact. Consider an open cover **F** of A without any finite subcover (such open cover exists as A is not compact) Since A is totally bounded we can cover A by finitely many open balls of radius 1. Then for atleat one of these open balls of radius 1, say $B_0 = B(x_0, 1)$, there is no finite subcover from the given open cover **F**. As $B_0 \subset A$, then B_0 is totally bounded. AS before, there is some $x_1 \in B_0$ such that $B_1 = B(x_1, \frac{1}{2})$ has no finite subcover from the given open cover **F**. In this way, proceeding inductively we have a sequence (x_n) in A with $x_{n+1} \in B_n = B(x_n, \frac{1}{2^n})$ such that B_n has no finite subcover from the given open cover **F**. Since $d(x_n, x_m) \leq \sum_{j=n}^{m-1} d(x_j, x_{j+1}) \leq \sum_{j=n}^{m-1} \frac{1}{2^j} \leq \frac{1}{2^{n-1}}$ for all $m > n$, we see that (x_n) is a Cauchy sequence of A. As, A is complete, there exists $x \in A$ such that $x_n \to x$. Now there is one open set O in **F**, so that $x \in O$. Let $r > 0$ be such that $B = B(X, r) \subset O$ and choose n so large that $d(x, x_n) < \frac{r}{2}$ where $r > \frac{1}{2^{n-1}}$. Now If$b \in A$ and $d(x_n, b) < \frac{1}{2^n}$, then

$d(x, b) < r$, so that $B_n \subset B \subset O$. Thus O is a finite subcover of B_n in the given open cover $\mathbf{F}$, in contradiction to our construction of B_n. Hence A is compact. □

Corollary 2.15. (1) ***(Heine-Borel)*** *A subset of a Euclidean space is compact if and only if it is closed and bounded.*
(2) ***(Bolzano-Weierstrass)*** *Every bounded sequence in a Euclidean space has a convergent subsequence.*

Corollary 2.16. *Let X be a complete metric space and $Y \subseteq X$.*
(1) *Y is compact if and only if Y is closed and totally bounded.*
(2) *$\bar{Y}$ is compact if and only if Y is totally bounded*

We also have the following easy fact:

Theorem 2.17. *Every totally bounded metric space (and in particular every compact metric space) is separable.*

Proof. If X is totally bounded, then there exists for each n a finite subset $A_n \subseteq X$ such that, for every $x \in X, d(x, A_n) < \frac{1}{n}$. Now let $A = \bigcup_{n=1}^{\infty} A_n$. The set A is either finite or countably infinite (why?); and for each $x \in X$ we have $d(x, A) \leq d(x, A_n) < \frac{1}{n}$, hence $d(x, A) = 0$, hence $x \in \bar{A}$ (why?). This proves that A is dense in X. □

Remark 2.18. Intuitively, a separable space is one that is "well approximated by a countable subset", while a compact space is one that is "well approximated by a finite subset".

Next we will give some simple results which will show how compact sets reproduce many other compact sets.

Theorem 2.19. *In a metric space (X, d),*
(1) *union of finite number of compact sets is compact.*
(2) *intersection of any collection of compact sets is compact*
(3) *sum of two compact sets is compact*

Proof. Proof of the above results are very simple and left to the students as exercise. □

Remark 2.20. An infinite union of compact sets need not even be closed (give an example!); and even when it is closed, it need not be compact (give another example!).

3. Continuous functions on compact metric spaces

From the definition of continuity, it is clear that inverse image of a closed set is closed but image of a closed set in geneal need not be closed. However, for some special closed sets namely, the compact ones — the direct image is closed and indeed is compact:

Proposition 3.1. (Continuous image of a compact space) *Let X and Y be metric spaces, with X compact, and let $f : X \to Y$ be a continuous map that is surjective (i.e. the image $f(X)$ equals all of Y). Then Y is compact.*

Proof. **FirstProof** : Let $(U_\alpha)_{\alpha \in I}$ be an open covering of Y . Then the sets $f^{-1}(U_\alpha)$ are open for each $\alpha \in I$, as f is continuous. Hence, $f^{-1}(U_\alpha)_{\alpha \in I}$ form an open covering of X (as f is surjective). Since X is compact, there exists a finite subset $J \subseteq I$ such that $f^{-1}(U_\alpha)_{\alpha \in J}$ still forms a covering of X. But then $(U_\alpha)_{\alpha \in J}$ forms a covering of Y .
SecondProof : Consider a sequence (y_n) of elements of Y . Because f is surjective, we can choose a sequence (x_n) of points in X such that $f(x_n) = y_n$. Since X is compact, there exists a subsequence (x_{n_k}) that converges to some point $a \in X$. But since f is continuous at a, the sequence (y_{n_k}) converges to $f(a)$. This proves that Y is sequentially compact, hence compact. □

This can be formulated precisely in several slightly different, but equivalent, ways:
(1) *Let X and Y be metric spaces, with X compact, and let $f : X \to Y$ be a continuous map. Then $f(X)$ is a compact subset of Y .*
(2) *Let X and Y be metric spaces, and let $f : X \to Y$ be a continuous map. If A is a compact subset of X, then $f(A)$ is a compact subset of Y .*
You should make sure you understand why these three formulations are equivalent.
It follows easily from the above Proposition that a continuous real-valued function on a compact metric space is automatically bounded, and furthermore that the maximum and minimum values are attained:

Corollary 3.2. *Let X be a compact metric space, and let $f : X \to \mathbb{R}$ be continuous. Then $f(X)$ is bounded, and there exist points $a, b \in X$ such that $f(a) = inf_{x \in X} f(x)$ and $f(b) = sup_{x \in X} f(x)$.*

Proof. As X is compact, $f(X)$ is a compact subset of $\mathbb{R}$, hence closed and bounded. Now, any bounded set $A \subseteq \mathbb{R}$ has a least upper bound $supA$ and a greatest lower bound $infA$, and these two points belong to the $\bar{A}$. But applying this to $A = f(X)$, which is closed, we conclude that $supf(X)$ and $inff(X)$ belong to $f(X)$ itself, which is exactly what is being claimed. □

Remark 3.3. This result can fail if X is noncompact, for instance if $X = \mathbb{R}$: a continuous real-valued function on $\mathbb{R}$ need not be bounded; and even if it is bounded, its supremum and infimum need not be attained. You should give examples to illustrate both these points.
In fact, you will learn in due time that a metric space is compact if and only if every continuous real-valued function on it is bounded.

We begin by recalling that if (X, d_X) and (Y, d_Y) are two metric spaces, then a mapping $f : X \to Y$ is continuous at the point $x \in X$ if, for each

$\epsilon > 0$, there exists > 0 (depending of course on ϵ) such that, for all $x \in X, d_X(x, x) <\Longrightarrow d_Y(f(x), f(x)) < \epsilon$. In particular, a mapping $f : X \to Y$ is continuous if it is continuous at every point $x \in X$, i.e. if, for each $\epsilon > 0$ and each $x \in X$, there exists > 0 (depending onϵ and x) such that, for all $x \in X, d_X(x, x) <\Longrightarrow d_Y(f(x), f(x)) < \epsilon$. Note that here can depend on x as well as on ϵ. We now make a new definition:

Definition 3.4. A mapping $f : X \to Y$ is uniformly continuous if, for each $\epsilon > 0$, there exists $\delta > 0$ (depending of course onϵ) such that, for all $x, x \in X$, $d_X(x, x) < \delta$ implies $d_Y(f(x), f(x)) < \epsilon$. The point is that, in uniform continuity, δ can still depend on ϵ (in general it has to) but is not allowed to depend on x.
To see that uniform continuity is truly a stronger property than continuity, consider the map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. It is continuous but not uniformly continuous. The existence on $\mathbb{R}$ of a function that is continuous but not uniformly continuous is directly linked to the fact that $\mathbb{R}$ is noncompact. In particular, we have:

Proposition 3.5. *Let f be a continuous mapping of a compact metric space X into a metric space Y . Then f is uniformly continuous.*

Proof. Given $\epsilon > 0$ and $x \in X$, there exists a $\delta_x > 0$ such that $d_X(x, x) < \delta_x$ implies $d_Y(f(x), f(x)) < \frac{\epsilon}{2}$. Now let $U_x = B(x, \frac{\delta_x}{2})$, the open ball of center x and radius $\frac{1}{2}\delta_x$. he collection $U_{x x \in X}$ is an open covering of X, so it has a finite subcovering $\{U_{x_1}, ..., U_{x_n}\}$. Let $\delta = \frac{1}{2} min \delta_{x_1}, ..., \delta_{x_1}. Clearly \delta > 0$. Now, given two points $y, z \in X$ such that $d_X(y, z) < \delta$, the point y must belong to some U_{x_i} and hence $d_X(y, x_i) < \frac{1}{2}\delta_{x_i}$. But then $d_X(z, x_i) \leq d_X(z, y) + d_X(y, x_i) < \delta + \frac{1}{2}\delta_{x_i} \leq \delta_{x_i}$.
So both y and z lie at a distance less than δ_{x_i} from the point x_i, which implies (by definition of $_{x_i}$) that $d_Y(f(y), f(x_i)) < \frac{\epsilon}{2}$ and $d_Y(f(z), f(x_i)) < \frac{\epsilon}{2}$. Hence $d_Y(f(y), f(z)) < \epsilon$, which shows that f is uniformly continuous. □

Remark 3.6. It is natural to ask whether the converse to this theorem is true: that is, if X is a metric space such that every continuous real-valued function on X is uniformly continuous, is X necessarily compact? The answer is no: for instance, if X is any discrete metric space, then every real-valued function on X is automatically both continuous and uniformly continuous; but a discrete metric space is compact if and only if it is finite.

Eercises:

(1) Determine whether the following subsets of $\mathbb{R}^2$ are compact:
   (a) $A = (\mathbb{Q} \cap [0, 1]) \times [0, 1]$
   (b) $B = \{(x, y) \in \mathbb{R}^2 : x = 0\}$
   (c) $C = (\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}) \times [0, 1]$
   (d) $D = \{(\frac{1}{n}, \frac{n-1}{n}) : n \in \mathbb{N}\}$
(2) Let A be closed and B a compact set in a metric space (X, d). Show that $A \cap B$ and B^d are compact in X.

(3) If every proper closed subset of a metric space (X, d) is finite, show that X is compact.

(4) Let A_1 be a compact subset of a metric space (X_1, d_1) and A_2 be a compact subset of a metric space (X_2, d_2). Show that $A_1 \times A_2$ is compact in $X_1 \times X_2$ (with product metric).

(5) Show that every totally bounded subset of a metric space is bounded.

(6) Consider the metric space l_2, with usual l_p metric d and the subset $A = \{x \in l_2 : d(x, \bar{O}) = 1\} \subset l_2$, where $\bar{O}$ is the zero sequence. Show that A is a bounded subset of l_2 but not totally bounded.

(7) Show that a subset of a totally bounded set of a metric space is totally bounded.

(8) Prove that in $\mathbb{R}$, a set A is bounded if and only if it is totally bounded.

(9) Show by an example that boundedness and total boundedness are not topological properties.

(10) Prove that a subset A of a metric space is X is totally bounded iff every sequence in A has a Cauchy subsequence.

(11) Let X be a nonempty compact subset of $\mathbb{R}^2$. Show that there exists $x_0 \in X$ such that $d(x_0, 0) = sup_{x \in X} d(x, 0)$
(12) Show that $A = \{f \in C[0, 1] : f(0) = 1\}$ is a closed subset of $C[0, 1]$
(13) Let (X, d_X) and (Y, d_Y) be metric spaces, where X is a compact space, and $f : X \to Y$ be a continuous bijection. If $f^{-1} : Y \to X$ is also continuous, then show that f is a homeomorphism.
(14) Given two nonempty disjoint sets, A, B in a metric space (X, d) with A compact and B clised, show that $d(A, B) > 0$.
(15) Let A, B be subsets of a metric space (X, d) with B compact. Prove that $d(A, B) = 0$ iff $\bar{A} \cap B \neq \phi$.
(16) Let A be a compact subset of a metric space (X, d). Prove that there exist $x, y \in A$ such that $d(x, y) = diam A$.
(17) Let f be a continuous real valued function on $[a, b]$. Prove that graph of f is a compact subset of $\mathbb{R}^2$.
(18) Prove that a metric space (X, d) is compact iff every real-valued continuous function on X is bounded.
(19) Let X and Y be two metric spaces. Show that a function $f : X \to Y$ is continuous iff the restriction of $f$to every compact subset of X is continuous.
(20) Show that the closed unit ball of $C[0, 1]$ is not compact.

Acknowledgments

This Class note is prepared from the books/ Lecture notes given to the references.

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