Concept of Degrees of Freedom

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1 Degrees-of-freedom of a mechanical system

Degree-of-freedom of a general mechanical system is defined as the *minimum* number of *independent* variables required to describe its configuration completely. The set of variables (dependent or independent) used to describe a system are termed as the *configuration variables*. For a mechanism, these can be either Cartesian coordinates of certain points on the mechanism, or the joint angles of the links, or a combination of both. The set of configuration variables form what is known as the *configuration space* (denoted by C) of the mechanism.

The degrees-of-freedom of a mechanical system (denoted by N) may or may not equal the dimension of C (denoted by dim(C)). Consider, e.g., a particle *free to move* in the XY plane. Clearly, the particle has two degrees-of-freedom, namely: the two independent translations in the plane. These can be completely described by the Cartesian coordinates (x,y), or the planar polar coordinates (r,φ), where:

$$x = r\cos\varphi$$
$$y = r\sin\varphi \qquad (1)$$

For such a *unconstrained system*, it is obvious that $N = \dim(C)$. However, mechanisms are typically *constrained mechanical systems*, and as such for them, the above equation takes the following form:

$$N = \dim(C) - n_c, \qquad (2)$$

where n_c is the number of *independent holonomic* constraints. These constraints can be represented as equations in the configuration variables q_j:

$$\eta_i(q_j) = 0, \quad i = 1,\ldots,n_c,\ j = 1,\ldots,\dim(C). \qquad (3)$$

The above can be compacted into a vector equation:

$$\eta(q) = 0 \qquad (4)$$

It is also important to note, that at least in some cases, it is possible to describe the same system as both constrained as unconstrained. For example, consider a small bead confined to a circular ring of radius *c*. The configuration space, C, in this case is trivially visualised as the circle. Since the circle is a planar entity, and the geometry of the circle is independent of the choice of its origin, path of the particle can be modelled as:

$$x^2 + y^2 - c^2 = 0 \tag{5}$$

In the above description admits two variables: (x,y), and hence dim(C) = 2. However, the particle has only *one* degree-of-freedom, which is along the ring itself. Therefore, in this case, $n_c = 2-1 = 1$, and Eq. (5) gives the functional form of the only constraint applicable.

The use of polar coordinates in the above case obviates the need for the constraint equation in the explicit form as above. In this case, the constraint can be absorbed in the description: $r = c$ (see Fig. 1). Therefore, the remaining variable φ becomes *free*, and therefore in this description of the problem, N = dim(C) = 1.

The choice of the constrained or the unconstrained description of a system is a matter of choice left to the analyst. It depends upon the objective of the analysis, ease of formulation and implementation (and also, personal taste!). In general, the constrained approach is more generic. While dim(C) tend to be higher in this case, the mathematical complexity of the constraint functions, η_i, tend to be lower. This approach is more amenable to numerical modelling and it is popular among computer tools for the simulation of general mechanical system, such as `ADAMS`. On the other hand, the second approach which tries to bring dim(C) down to N, is usually more cumbersome. It needs special treatment of each problem separately. However, if successful, the results are highly rewarding in terms of deeper understanding of the *intrinsic* characteristics of the system. This approach, is therefore, more attractive to researchers. Often, computer algebra systems (CAS) such as `Mathematica`, are employed to aid the calculations in this case.

2 Degrees-of-freedom of planar systems

In the following, we grow the system from just an isolated point to a planar manipulator in a sequence of steps, analysing each step individually as we proceed.

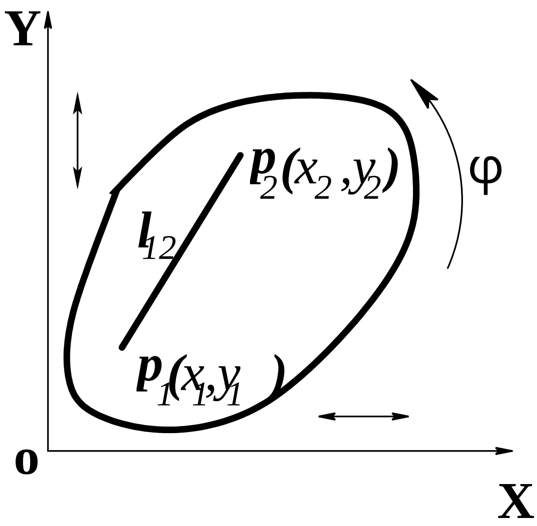

Figure 1: A line segment embedded in a planar rigid body

2.1 A line in the plane

A *rigid* line segment consists of two points on a rigid-body, as shown in the Fig. 1. In the plane, the configuration can be described by the coordinates of the two end-points, $p_1(x_1, y_1)$ and $p_2(x_2, y_2)$. Hence, a possible description of the system is the following:

$$q = (x_1, y_1, x_2, y_2)$$

$$\eta_1 = d\,(p_1, p_2) - l_{12} = 0$$

$$\Rightarrow (x_1 - x_2)^2 + (y_1 - y_2)^2 - l_{12}^2 = 0$$

Therefore, in the constrained description, dim(C) = 4, $n_c = 1$. From Eq. (2), we have $N = 4-1 = 3$. The unconstrained description of the system therefore admits three independent variables. In this case, these can be chosen as $q(x_1, y_1, \varphi)$ (or equivalently, as $q(x_2, y_2, \varphi)$). As mentioned earlier, this description affords easy physical interpretation: i.e., a line (segment) embedded in a planar rigid body can be described by prescribing the position of one point on the line, and the *orientation* of the line with respect to a fixed line (which, by default, is the X axis). Interestingly, these variables directly relate to the physical motions of the line, the translation along X, translation along Y , and CCW rotation about the Z axis respectively.

2.2 A rigid-body in a plane

A rigid-body confined to a plane has no more degrees-of-freedom than a line embedded in it. In other words, it is enough to specify the coordinates of two points on the body in order to describe its configuration completely. To illustrate this, let us introduce a third point, $p_3(x_3, y_3)$, which is *non-colinear* with the segment p_1p_2 (see Fig. 2). The combined degrees-of-freedom of the three

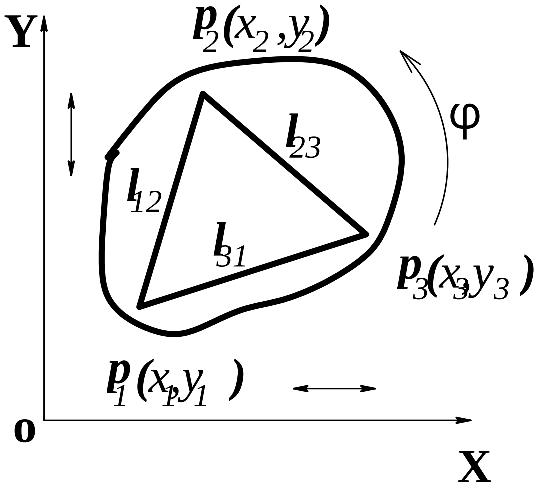

Figure 2: Rigid body in planar motion with three-degrees-of-freedom

points are dim(C) = 3 × 2 = 6, and $q = (x_1, y_1, x_2, y_2, x_3, y_3)^T$. Therefore, to justify our claim that $N = 3$, we must have $n_c = \dim(C) - N = 6 - 3 = 3$. It is not hard to find the constraint functions η_i, $i = 1,2,3$, in this case:

$$d(p_i, p_j) - l_{ij} = 0, \qquad i,j = 1,2,3,\ i = 6\ j. \tag{6}$$

While there are six distinct pairs of (i,j) in the above, only three matter (*why*?), and therefore our claim is verified. It may be noted here, that the actual distance of the points do not alter the above calculation, i.e., the values of l_{ij} are immaterial in the above so long as we understand that they are constants.

Further, it may be verified that if we continue this process with four or more points in the plane, we would see the degree-of-freedom boil down to the same number in each case. The system of constraints become more complicated, making it an interesting mental exercise which the reader is strongly encouraged to undergo.

2.3 A rigid link in the plane

A rigid binary link (see Fig. 3) is not different from a rigid line segment. Similarly, a ternary link is the same as a triangle rigidly embedded in a body. However, we mention it in the passing as these links form the building blocks of all mechanisms.

2.4 A system of links in the plane

In this case, we have two freely floating links in the plane as shown in Fig. 4 . We choose the unconstrained description as a starting point, which gives dim(C) = 2 × 3 = 6, and q = $(x_1, y_1, \varphi_1, x_2, y_2, \varphi_2)^T$ gives one possible set of configuration variables. Now, suppose we decide to

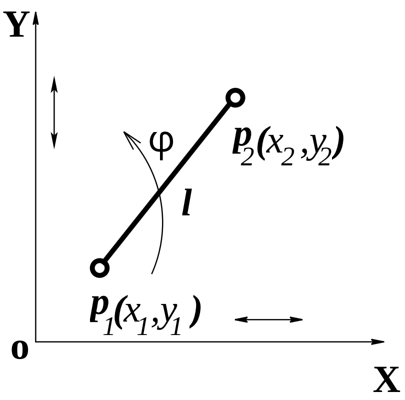

Figure 3: A rigid link in planar motion

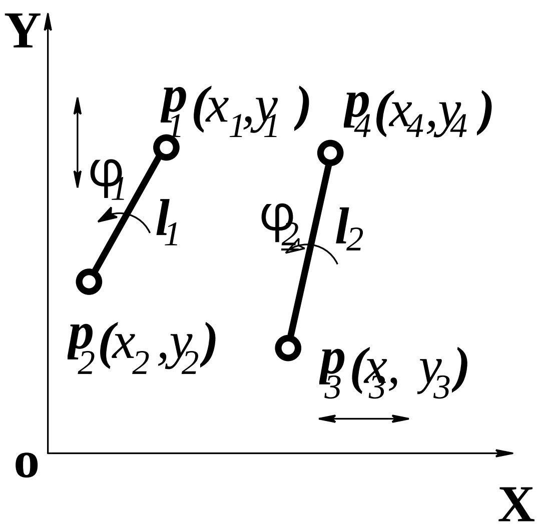

Figure 4: Two free links in planar motion

create an open chain of two links. This can be done in various ways, i.e., linking different pairs of the end points. We choose to mate the point p2 with the point p3, as shown in Fig. 5. This imposes

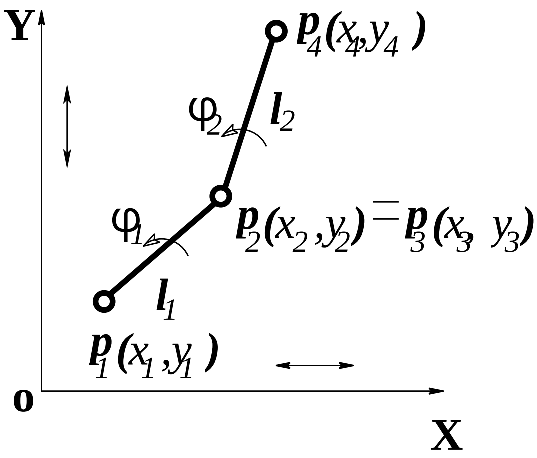

Figure 5: Two planar links with a rotary joint

two constraints:

$$x_3 - x_2 = 0$$

$$y_3 - y_2 = 0$$

Therefore, dim(C) = 6 − 2 = 4. It is convenient to think of these degrees-of-freedom as being reflected as the configuration of the lower link, and the relative rotation, $\theta_2 = \varphi_2 - \varphi_1$ of the upper link with respect to the lower one. Consider further, that the chain is now tethered down to a point, $p_g(x_g, y_g)$, on the plane as shown in Fig. 6. An additional pair of constraints, similar to the

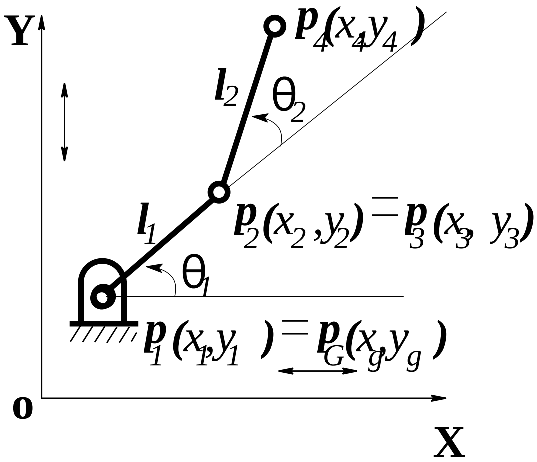

Figure 6: Planar 2R manipulator

one above, is imposed in the process:

$$x_1 - x_g = 0$$

$$y_1 - y_g = 0$$

The last pair of constraints essentially kill the translational degree-of-freedom of the lower link. Its remaining degree-of-freedom can be conveniently described by the joint angle _1 = _1. The above steps illustrate how one can choose among various options of representing the same system in terms of different sets of variables. In particular, in this case, starting from the Cartesian coordinates describing points and links, we ended up with an open-chain manipulator. Starting from a constrained description, we came up with an unconstrained description of the system in terms of the joint coordinates. As we shall see, the joint angles provide the most convenient as well as natural description for all open chains.

- **Example: Degrees of freedom of human arm using joint convention.**

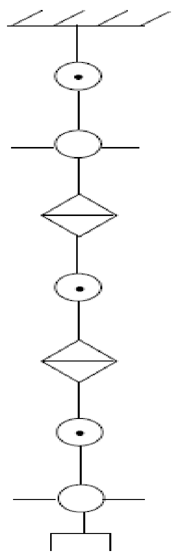

Fig. Seven joint human

- The human arm has three joint cluster in it, the shoulder, the elbow and the wrist.
- The shoulder has three degrees of freedom, since the upper arm (humerus) can rotate in the sagittal plane (parallel to the mid plane of the body), the caronal plane (a plane from shoulder to shoulder), and about the humerus.
- The elbow has only one degree of freedom; it can only flex and extend about the elbow joint.
- The wrist also has three degrees of freedom. It can abduct, flex and extend, and since the radius bone can role over the ulna bone, it can rotate longitudinally (pronate and supirate). Thus, the human arm has a total of seven degrees of freedom, even if the range of some movements are small.

(Courtesy: S.Bandyopadhyay & Md.K.Zaman)