If f be the acceleration at P,

$$f\cos\frac{\pi}{4} = f_t \text{ and } f\sin\frac{\pi}{4} = f_n$$

Thus $\sqrt{2}\,f_t = \sqrt{2}\,f_n = f$ (4)

Now, $f_t = f_n$ gives

$$\frac{dv}{dt} = \frac{v^2}{\rho}$$

or, $$\frac{d}{dt}\left(c\sec^2\frac{d\psi}{dt}\right) = c\sec^2\psi\left(\frac{d\psi}{dt}\right)^2$$ [using (2) and (3)]

or, $$\frac{d}{dt}\left(\sec^2\psi\frac{d\psi}{dt}\right) = \left(\sec^2\psi\frac{d\psi}{dt}\right)\frac{d\psi}{dt}$$

or, $$\frac{d\left(\sec^2\psi\frac{d\psi}{dt}\right)}{\sec^2\psi\frac{d\psi}{dt}} = d\psi$$

Integrating, $$\log\left(\sec^2\psi\frac{d\psi}{dt}\right) = \psi + A \qquad \ldots (5)$$

where A is constant of integration.

At $t = 0, \psi = 0$ and $v = u$

By (2), $u = c\left(\frac{d\psi}{dt}\right)_0$ i.e., $\left(\frac{d\psi}{dt}\right)_0 = \frac{u}{c}$

So from (5), $$\log\left(\frac{c}{u}.\sec^2\psi\frac{d\psi}{dt}\right) = \psi$$

or, $$c\sec^2\psi\frac{d\psi}{dt} = ue^{\psi}$$

So, by (2) velocity $v = ue^{\psi}$

Again from (4), acceleration $f = \sqrt{2}.f_n$

or, $$f = \sqrt{2}.c\left(\sec^2\psi\frac{d\psi}{dt}\right)\frac{d\psi}{dt}$$

$$= v.\frac{1}{\rho}.v \qquad [\rho = \text{radius of curvature} = \frac{ds}{d\psi}].$$

$$= \frac{v^2}{\rho}$$

This acceleration is along the inward drawn normal.

Particular case : *A particle moves in a circle of radius a with uniform speed v.*

For a circle of radius a, $s = a\psi$

so $v = \frac{ds}{dt} = a\frac{d\psi}{dt} = \text{constant}$ [v is uniform]

The tangential acceleration $= \frac{d^2s}{dt^2} = 0$

and the normal acceleration $= \frac{v^2}{a}$, since $\rho = a$.

Thus the *acceleration is entirely along normal* towards the centre of the circle. This acceleration is called *centripetal acceleration.*

Again, if the angular velocity of the particle is w, $v = aw$

and the normal acceleration is $\frac{v^2}{a} = \frac{a^2w^2}{a} = aw^2$

If θ be the angle at the centre made by the radius vector with the initial line; $\frac{d\theta}{dt} = w$

and $v = aw = a\frac{d\theta}{dt}$.

If v is not uniform,

$$f_t = a\frac{d^2\theta}{dt^2} \text{ and } f_n = a\left(\frac{d\theta}{dt}\right)^2.$$

9.2. Motion of a particle under gravity along a smooth curve in a vertical plane.

Let a particle of mass m be moving along a given smooth curve APB in the x-y plane; where y-axis is in vertically upward direction.

Solution : Let $P(x,y)$ be the position of the particle at any time t.

Then $\frac{dx}{dt}=\lambda y$ and $\frac{dy}{dt}=\mu$,

[where λ and μ are constants]

$$\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}=\frac{\mu}{\lambda y}$$

or, $\mu dx=\lambda y dy$

Integrating, $\mu x=\frac{1}{2}\lambda y^2+A$

[A being a constant of integration.]

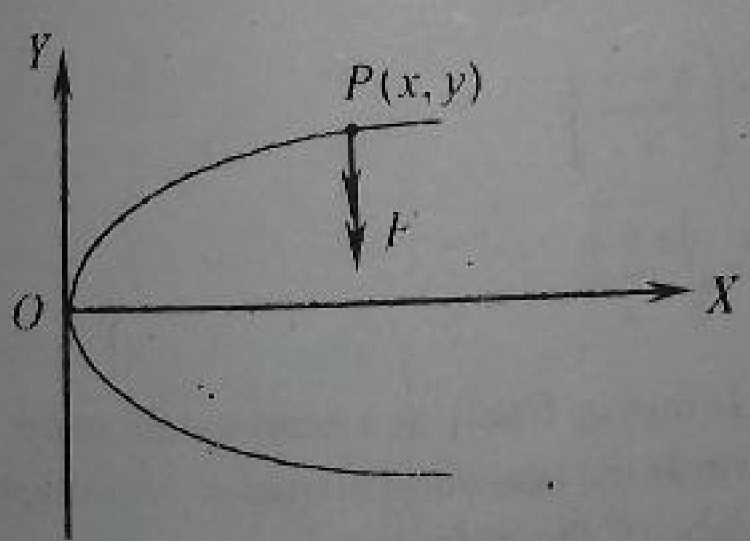
or, $y^2=\frac{2\mu}{\lambda}\left(x-\frac{A}{\mu}\right)$

which represents a parabola, whose axis is along the x-axis, vertex at $\left(\frac{A}{\mu},0\right)$ and latus rectum $\frac{2\mu}{\lambda}$.

Ex. 4. A particle describes a parabolic path moving under a force which is always perpendicular to its axis. Find the law of force and the velocity of the particle at any point in the orbit. [*C. P. 1982*]

Solution : Without any loss of generality, we take the equation of the parabola as $y^2=4ax$, ... (1)

whose vertex is at the origin and axis along the x-axis.



Equations of motion of the particle are

$$\ddot{x}=0 \quad (2)$$

and, $$\ddot{y}=-F \quad (3)$$

From (1) differentiating with respect to t

$$\frac{x\dot{x}}{a^2}+\frac{y\dot{y}}{b^2}=0 \qquad \dots (4)$$

Multiplying (2) and (3) by $\frac{\dot{x}}{a^2}$ and $\frac{\dot{y}}{b^2}$ respectively and adding the products

$$\frac{\ddot{x}\dot{x}}{a^2}+\frac{\ddot{y}\dot{y}}{b^2}=-\frac{F}{r}\left(\frac{x\dot{x}}{a^2}+\frac{y\dot{y}}{b^2}\right)=0 \qquad \text{[using (4)]}$$

Integrating with respect to t

$$\frac{\dot{x}^2}{a^2}+\frac{\dot{y}^2}{b^2}=\text{constant}=\mu\text{, say} \qquad \dots (5)$$

Again, differentiating (4) with respect to t, we have

$$\frac{x\ddot{x}}{a^2}+\frac{y\ddot{y}}{b^2}+\frac{\dot{x}^2}{a^2}+\frac{\dot{y}^2}{b^2}=0$$

Putting the values of $\ddot{x}$ and $\ddot{y}$ from (2) and (3) respectively and using (5),

$$-\frac{F}{r}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)+\mu=0\text{,}$$

or, $-\frac{F}{r}+\mu=0$ [using (1)]

i.e., $F=\mu r$, i.e., $F \propto r$.

8.12. Illustrative Examples

Ex. 1. The position of a moving particle at any time t is given by $x=a\cos t$, $y=a\sin t$; find its path, velocity and acceleration.

Solution : Here, $x=a\cos t$, $y=a\sin t$.

The equation to the path is obtained by eliminating t between the expressions for x and y.

Thus, the equation of the path is

$$x^2+y^2=a^2(\cos^2 t+\sin^2 t)=a^2$$

which represents a *circle of radius a*.

or, $\dfrac{\dot{s}^2}{\dfrac{ds}{dt}\cdot\dfrac{dt}{d\psi}} = b$, i.e., $\dot{s}\dfrac{d\psi}{dt} = b$

Using (1) in (2), $(at + c)\dfrac{d\psi}{dt} = b$

or, $d\psi = \dfrac{b\,dt}{at + c}$

Integrating $\psi = \dfrac{b}{a}\log(at + c) + \log K$

Assuming $\psi = 0$ when $t = 0$, $K = -\dfrac{b}{a}\log C$

or, $\psi = A\log(1 + Bt)$,

where $A = \dfrac{b}{a}$ and $B = \dfrac{a}{c}$ are constants.

Ex. 8. A particle is describing a circle of radius a in such a way ... tangential acceleration is K times the normal acceleration, where ... constant. If the speed of the particle at any point be u, prove that ... return to the same point after a time $\dfrac{a}{Ku}\left(1 - e^{-2\pi K}\right)$

[C.P. 1994, 1998, 2000, 2003, 2005; ...

Solution : When a particle moves in a circle of radius a with angular velocity w,

Velocity $v = aw = a\dfrac{d\theta}{dt}$

normal acceleration $= \dfrac{v^2}{\rho} = \dfrac{a^2\left(\dfrac{d\theta}{dt}\right)^2}{a} = a\left(\dfrac{d\theta}{dt}\right)^2$

and tangential acceleration $= \dfrac{dv}{dt} = a\dfrac{d^2\theta}{dt^2}$.

Here, $a\dfrac{d^2\theta}{dt^2} = Ka\left(\dfrac{d\theta}{dt}\right)^2$

Again, we have, $\dot{x} = \frac{dx}{dt} = -a\sin t$, $\dot{y} = \frac{dy}{dt} = a\cos t$

If v be the velocity,

$$v^2 = \dot{x}^2 + \dot{y}^2 = (-a\sin t)^2 + (a\cos t)^2 = a^2$$

So, $v = a$, making an angle $\tan^{-1}\left(\frac{a\cos t}{-a\sin t}\right)$

$$= \tan^{-1}(-\cot t) = \tan^{-1}\left\{\tan\left(\frac{\pi}{2}+t\right)\right\} = \frac{\pi}{2}+t \text{ with the } x\text{-axis}$$

Also, $\ddot{x} = -a\cos t$, $\ddot{y} = -a\sin t$

Accleration $f = \sqrt{\ddot{x}^2 + \ddot{y}^2} = a$, in a direction making an angle

$\tan^{-1}\left(\frac{-a\sin t}{-a\cos t}\right) = \tan^{-1}(\tan t) = t$ with the x-axis.

Ex. 2. A particle moves in a plane, its velocities parallel to the axes of x and y are $u + \omega_1 y$ and $v + \omega_2 x$ respectively, where u, v, ω_1, ω_2 are constants. Show that the path of the particle is a conic section.

[*C. P. 1982, '84, 86, 92, B.P. 1998, V.P. 19..*

Solution : Here, $\frac{dx}{dt} = u + \omega_1 y$, $\frac{dy}{dt} = v + \omega_2 x$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{v + \omega_2 x}{u + \omega_1 y}$$

or, $(v + \omega_2 x)\,dx = (u + \omega_1 y)\,dy$

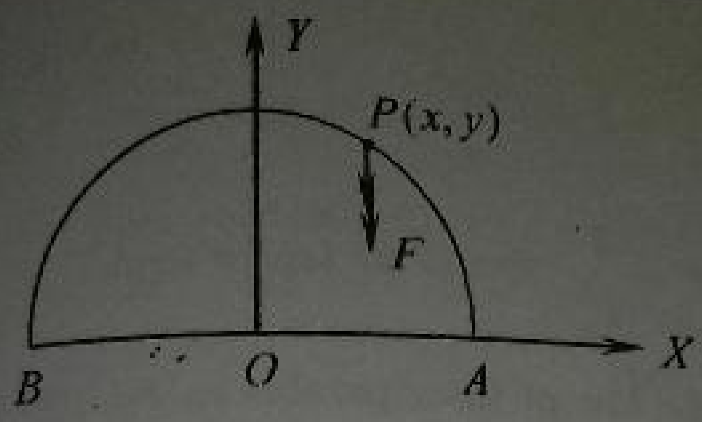
Integrating, $vx + \frac{1}{2}\omega_2 x^2 = uy + \frac{1}{2}\omega_1 y^2 + C$

[where C is a constant]

or, $2(vx - uy) + (\omega_2 x^2 - \omega_1 y^2) = 2C$

which represents a conic section.

Ex. 3. A particle is moving with a constant velocity parallel to the axis of y and velocity proportional to y parallel to the axis of x. Show that the path of the particle is a parabola.



F is the force directed perpendicularly to the bounding diameter AB of the semi-circle.

Equations of motion of the particle are

$$m\frac{d^2x}{dt^2}=0 \quad \ldots (1)$$

and, $$m\frac{d^2y}{dt^2}=-F \quad \ldots (2)$$

From (1), $\frac{dx}{dt}=\text{constant}=K$ (say)

Again, since $x^2+y^2=a^2$

$$2y\frac{dy}{dt}=-2x\frac{dx}{dt}=-2Kx$$

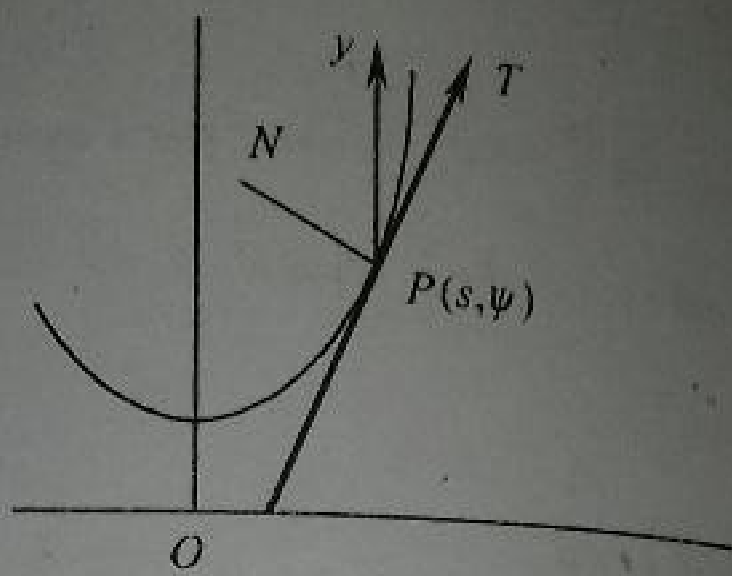
or, $$\frac{dy}{dt}=-\frac{Kx}{y}$$

So. $$\frac{d^2y}{dt^2}=-K.\left(\frac{y\frac{dx}{dt}-x\frac{dy}{dt}}{y^2}\right)$$

$$=-K\left(\frac{yK+x.\frac{Kx}{y}}{y^2}\right)$$

$$=-K.\frac{K(x^2+y^2)}{y^2.y}=-\frac{K^2a^2}{y^3}$$

Solution : Let $P(s,\psi)$ be the position of the particle at any time t. Given that its acceleration f makes equal angles $\frac{\pi}{4}$ with the tangent PT and the normal PN to the curve at P.



The equation of the path is

$$s = c.\tan\psi \quad \cdots (1)$$

Velocity at any point P

$$= v = \frac{ds}{dt} = c.\sec^2\psi\frac{d\psi}{dt} \quad \cdots (2)$$

and $\rho = \frac{ds}{d\psi} = c.\sec^2\psi$

Tangential acceleration $f_t = \frac{d^2s}{dt^2} = \frac{dv}{dt}$

$$= 2c.\sec\psi.\sec\psi\tan\psi\frac{d\psi}{dt}.\frac{d\psi}{dt} + c.\sec^2\psi\frac{d^2\psi}{dt^2}$$

$$= 2c.\sec^2\psi\tan\psi\left(\frac{d\psi}{dt}\right)^2 + c.\sec^2\psi\frac{d^2\psi}{dt^2}.$$

Normal acceleration

$$f_n = \frac{v^2}{\rho} = \frac{c^2\sec^4\psi\left(\frac{d\psi}{dt}\right)^2}{c\sec^2\psi} = c\sec^2\psi\left(\frac{d\psi}{dt}\right)^2 \quad \cdots (3)$$

9 Motion in a Plane : Tangential and Normal Acceleration

Expressions for the tangential and normal components of the acceleration of a particle describing a plane curve

A particle moves along the curve APQ and P, Q are its position on the curve at time t and $(t+\delta t)$ respectively, such that arc $AP = s$ and arc $AQ = s+\delta s$, i.e., arc $PQ = \delta s$.

Also, let v and $(v+\delta v)$ be the velocities of the particle at P and Q along the directions of the tangents to the curve at P and Q respectively.

The tangents at P and Q make angles ψ and $(\psi+\delta\psi)$ with a fixed line OX in the plane of the curve; then $\delta\psi$ is the angle between the tangents at P and Q.

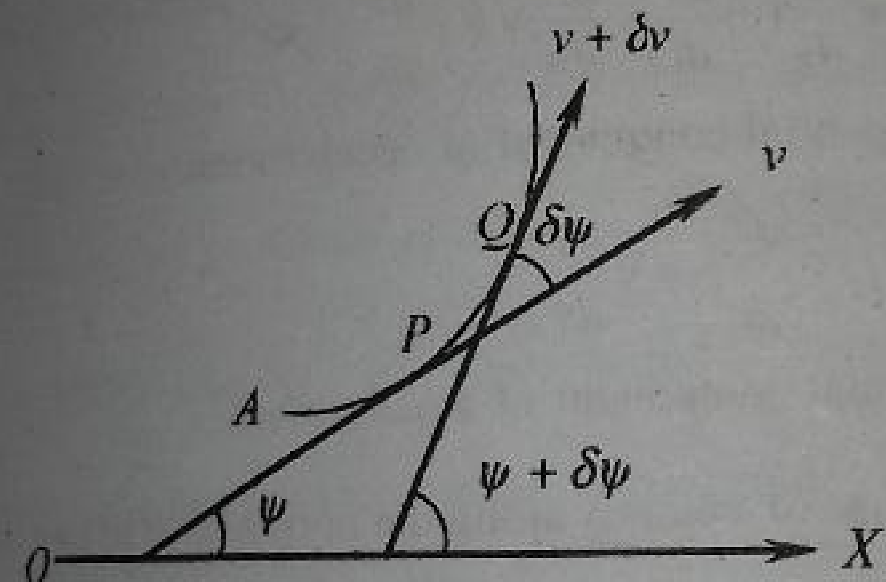


Fig. 9.1.1

The velocity v of the particle at P is given by

$$v = \frac{ds}{dt}$$

This velocity is along the tangent to the curve at P.

The acceleration along the tangent at P given by

$$f_t = \lim_{\delta t \to 0} \frac{\text{Velocity along the tangent in time}(t+\delta t) - \text{the same at time } t}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(v+\delta v)\cos\delta\psi - v}{\delta t}$$

If F be the force per unit mass acting on the particle in a direction parallel to the y-axis, the equations of motion are

$$\frac{d^2x}{dt^2}=0 \qquad \ldots (2)$$

and $$\frac{d^2y}{dt^2}=F \qquad \ldots (3)$$

From (2), $\frac{dx}{dt}=\text{constant}=u$, (say)

From (1), $$\frac{dy}{dt}=\frac{1}{2}.c-\frac{1}{2}\left(e^{\frac{x}{2}}-e^{-\frac{x}{2}}\right)\frac{dx}{dt}$$

$$=\frac{cu}{4}\left(e^{\frac{x}{2}}-e^{-\frac{x}{2}}\right)$$

and, $$\frac{d^2y}{dt^2}=\frac{cu}{4}.\frac{1}{2}\left(e^{\frac{x}{2}}+e^{-\frac{x}{2}}\right)\frac{dx}{dt}$$

$$=\frac{cu}{4}.\frac{1}{2}\left(e^{\frac{x}{2}}+e^{-\frac{x}{2}}\right).u$$

or, $$\frac{d^2y}{dt^2}=\frac{u^2}{4}.y \qquad | \text{ using } (1) |$$

Hence $$F=\frac{d^2y}{dt^2}=\frac{u^2}{4}.y$$

and $F \propto y$, this is the law of force.

Ex. 9. A particle describes the catenary $y=c.\cosh\frac{x}{c}$ under a force which is always parallel to the positive direction of the y-axis. Find the law of force and the velocity of the particle at any point on the path.

| *V. P. 1989, C. P. 1988, 2008* |