Closure properties of Regular Languages :-

If certain languages are regular, and a language L is formed from them by certain operations (Ex. L is the union of two regular languages), then L is also Regular.

So, the class of Regular Languages are closed under the operations mentioned below —

i) The Union of two Regular languages is Regular.

ii) The Intersection of two Regular Languages is Regular.

iii) The complement of a Regular Language is Regular.

iv) The Difference of two Regular languages is Regular.

v) The Reversal of a Regular Language is Regular.

vi) The Closure of a Regular Language is Regular.

vii) The Concatenation of Regular Languages is Regular.

viii) A Homomorphism (substitution of strings for symbols) of a Regular Language is Regular.

ix) The Inverse Homomorphism of a Regular Language is Regular.

* The Family of Regular Language is closed under Union, Concatenation and Star-closure

— If L_1 and L_2 are regular, then there exist regular expression r_1 and r_2 such that,

$$L_1 = L(r_1) \text{ and } L_2 = L(r_2)$$

But we know, when r_1 and r_2 are regular expression then,

$r_1 + r_2$, $r_1 \cdot r_2$ and r_1^* are also regular expression denoting languages $L_1 \cup L_2$, $L_1 L_2$ and L_1^* respectively.

So, they are closed under union, concatenation and star-closure.

Hence proved.

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* If L is regular language over alphabet Σ, then $\bar{L} = \Sigma^* - L$ is also regular.

— To show the closure under complement let, $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA that accepts L. Then DFA

$$\bar{M} = (Q, \Sigma, \delta, q_0, Q - F) \text{ accepts } \bar{L}.$$

That is $\bar{M}$ is exactly like M, but the accepting/final states of M become non-accepting states of $\bar{M}$ and vice-versa.

Then a string w is in $L(\bar{M})$ iff $\hat{\delta}(q_0, w)$ is in $Q - F$, which occures iff w is not in $L(M)$.

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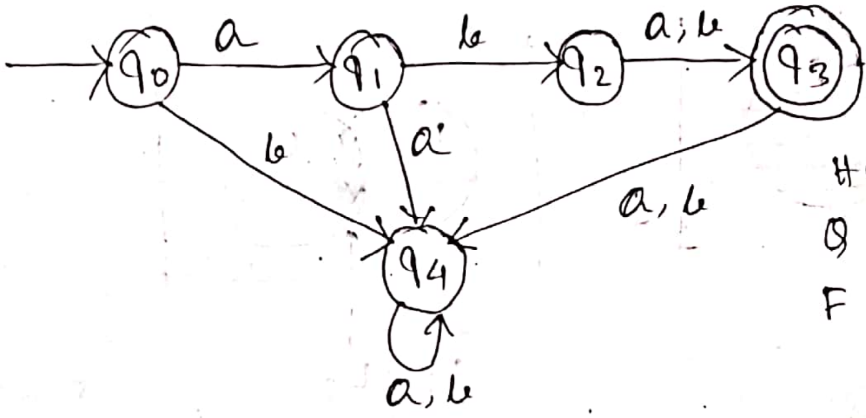
Ex An FA that accepts only strings aba and abb is shown in the figure. Find FA for the complement of L, i.e. accept every string other than aba and abb.

— The FA which accepts only aba and abb is – $M = (Q, \Sigma, \delta, q_0, F)$



Here,
$Q = \{q_0, q_1, q_2, q_3, q_4\}$
$F = \{q_3\}$

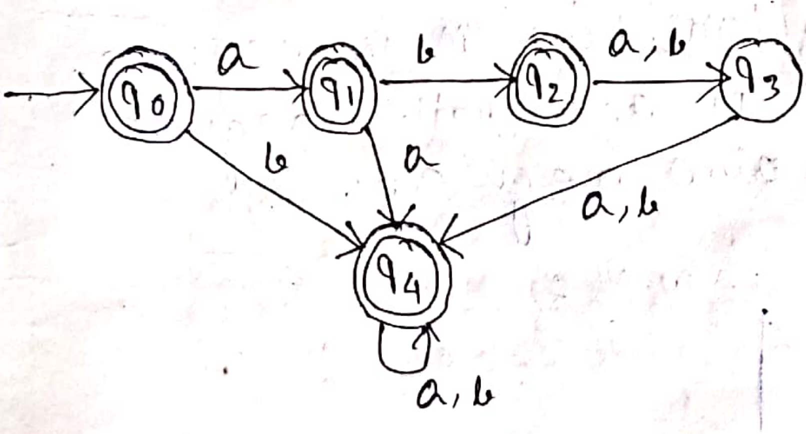
Now, applying the theorem Closure under complement,

$\overline{M} = (Q, \Sigma, \delta, q_2 ; Q - F)$

$\therefore \quad Q = \{q_0, q_1, q_2, q_3, q_4\}$

$Q - F = \{q_0, q_1, q_2, q_3, q_4\} - \{q_3\}$
$\quad = \{q_0, q_1, q_2, q_4\}$

$\therefore$ The FA is,

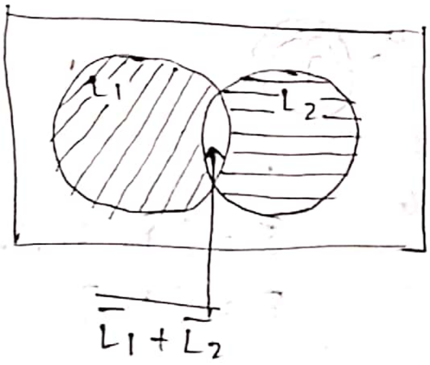


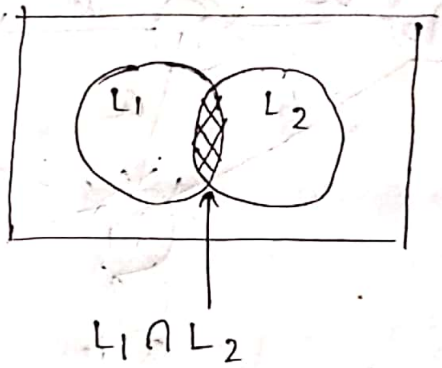
It accepts all the strings except aba and abb.

* The set of Regular Languages is closed under Intersection.

— Let, L_1 and L_2 are regular languag.
Then, according to D'morgans Theorem,

$$L_1 \cap L_2 = \overline{\overline{L_1} + \overline{L_2}}$$





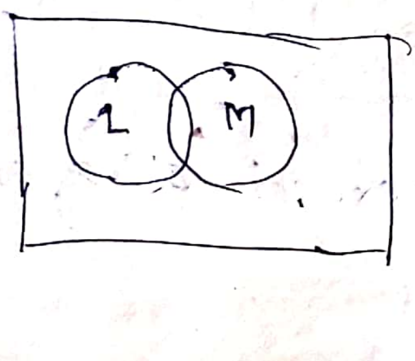
Now, L_1 and L_2 are regular, so $\overline{L_1}$ and $\overline{L_2}$ is also regular, and so $\overline{L_1} + \overline{L_2}$ and so $\overline{\overline{L_1} + \overline{L_2}}$ is also regular.
which implies $L_1 \cap L_2$ is regular.

* If L and M ~~and~~ are regular the L-M is also regular.

— We know,

$$L - M = L \cap \overline{M}.$$

Here, $\overline{M}$ is regular, bcoz M is regular.
so, when L and $\overline{M}$ are both regular the $L \cap \overline{M}$ is also regular.



Reversal

The reversal of a string $a_1 a_2 \cdots a_n$ is the string written backwards, that is $a_n a_{n-1} \cdots a_2 a_1$. We use w^R for the reversal of string w.

Ex – $(0010)^R = (0100)$

$\Lambda^R = \Lambda$

If L is a language then, the reverse is L^R.

Let, $L = \{001, 10, 111\}$

$\therefore L^R = \{100, 01, 111\}$

Automata for L^R

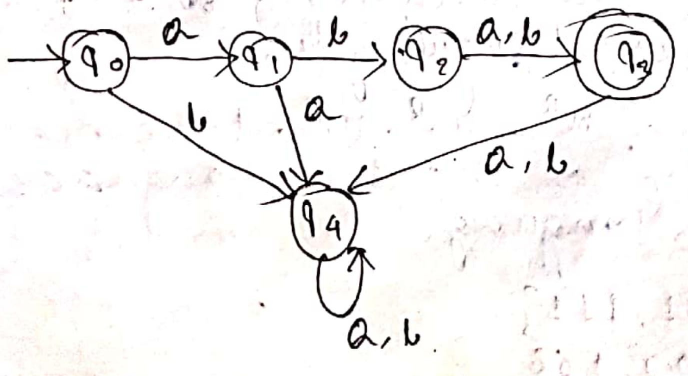
Given a language L, the reversal of L, which is L^R. Now, with Nondeterminism and Λ-transition we can construct an automaton for L^R from $L(M)$ –

i) Reverse all the arcs in the transition diagram for M.

ii) Make the start state of M be the only accepting state for the new automaton.

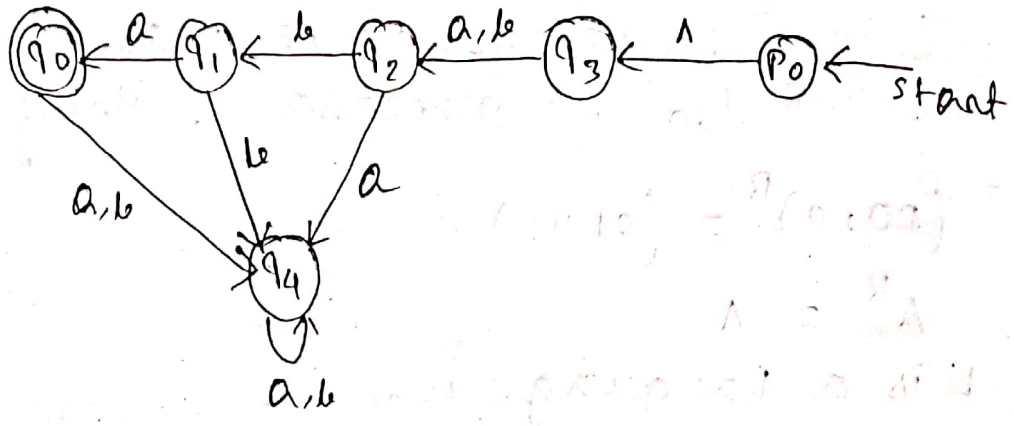
iii) Create a new start state p_0 with transition on Λ to all the accepting states of M.

Ex Let, $L(M)$ = Accepts only aba and abb.



So, L^R will accepts only ~~aaa~~ aba and bba.

∴ Automaton for L^R is –



The above automaton is reverse of $L(m)$. It accepts only {aba, bba}.

---

* If L is regular the L^R is also regular.

Let us assume that L is defined by the regular expression r. Now, we show that there is another regular expression r^R, such that $L(r^R) = (L(r))^R$, that is the language of r^R is the reversal of the language of r.

Now, If r is Λ, ϕ, or a for some symbol a, then r^R is same as r. bcoz,

$\{\Lambda\}^R = \{\Lambda\}$, $\{\phi\}^R = \{\phi\}$ and $\{a\}^R = \{a\}$

Now, there are three cases depending on the forms of the r,

i) $r = r_1 + r_2$. Then $r^R = r_1^R + r_2^R$. The justification is that the reversal of the union of two languages is obtained by computing the reversal of the two languages and taking the union of those languages.

ii) $r = r_1 . r_2$. Then $r^R = r_2^R . r_1^R$. Here we reverse the order of the two languages, as reversing the language themselves.

Ex- Let, $L(r_1) = \{01, 111\}$

$L(r_2) = \{00, 10\}$

then $L(r_1 r_2) = \{0100, 0110, 11100, 11110\}$

Now, $(L(r_1 r_2))^R = \{0010, 0110, 00111, 01111\}$

$= (L(r_2))^R \cdot (L(r_1))^R$

$= \{00, 01\}\{10, 111\}$

$= \{0010, 00111, 0110, 01111\}$

$= (L(r_1 r_2))^R$

iii) $r = r_1^*$. Then $r^R = (r_1^R)^*$. The justification is that any string w in $L(R)$ can be written as $w_1; w_2, \ldots, w_n$, where each w_i is in $L(r)$.

But, $w^R = w_n^R \cdot w_{n-1}^R \cdots w_1^R$

each w_i^R is in $(L(r))^R$.

So, w^R is in $(r_1^R)^*$

Homomorphism

A string homomorphism is a function on strings that works by substituting a particular string for each symbol.

Let, Σ and Σ' are alphabets. Then the function,

$$h : \Sigma \longrightarrow \Sigma' \text{ is called a homomorphism.}$$

If, $w = x_1 x_2 \cdots x_n$ then,

$$h(w) = h(x_1)\, h(x_2) \cdots h(x_n)$$

If L is a language on Σ, then its homomorphic image is defined as,

$$h(L) = \{h(w) : w \in L\}$$

Ex Let $\Sigma = \{0, 1\}$ and $\Sigma' = \{0, 1, 2\}$ and defined h by, $h(0) = 01$, $h(1) = 112$. Then find $h(010)$ and homomorphic image of $L = \{00, 010\}$.

— Given, $\Sigma = \{0, 1\}$ and $\Sigma' = \{0, 1, 2\}$

Now, $h(0) = 01$, $h(1) = 112$.

So, $h(010) = 0111201$

The homomorphic image of $L = \{00, 010\}$ is the language,

$$h(L) = \{0101, 0111201\}$$

---

* Let h be a homomorphism. If L is a regular, then its homomorphic image $h(L)$ is also regular. The family of regular languages is closed under arbitrary homomorphisms.

— Let $L = L(R)$ for some regular expression R. In general if r is a regular expression with symbol Σ, let $h(r)$ be the expression we obtain by replacing each symbol a of Σ in r by $h(a)$. we claim that $h(R)$ define the language $h(L)$.

Now, if r is Λ or ϕ, then $h(r)$ is same as r, since h doesnot affect the string Λ or the language ϕ. Thus $L(h(r)) = L(r)$. However if r is ϕ or Λ, then $L(r)$ contains either no strings or a string with no symbol respectively. Thus $h(L(r)) = L(r)$ in either case. we conclude $L(h(r)) = L(r) = h(L(r))$.

In other case if $r = a$ for some symbol a in Σ, $L(r) = \{a\}$, so $h(L(r)) = \{h(a)\}$. Also, $h(r)$ is the regular expression that is the strings of symbols $h(a)$. Thus $L(h(r))$ is also $\{h(a)\}$, and we conclude, $L(h(r)) = h(L(r))$.

There are three cases –

i) let $r = r_1 + r_2$

$\therefore h(r) = h(r_1 + r_2) = h(r_1) + h(r_2)$.

we know, $L(r) = L(r_1) \cup L(r_2)$ and

$L(h(r)) = L(h(r_1) + h(r_2)) = L(h(r_1)) \cup L(h(r_2))$ ---- (a)

Again, h is applied to a language by application to each of its string individually. So we can write,

$$h(L(R)) = h(L(R_1) \cup L(R_2)) = h(L(R_1)) \cup h(L(R_2)) \quad \text{---} (b)$$

But, we know according to induction hypothesis,

$$L(h(R_1)) = h(L(R_1)) \quad \text{and}$$
$$L(h(R_2)) = h(L(R_2))$$

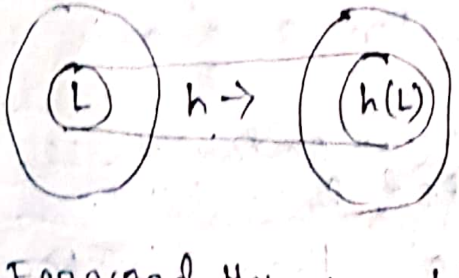
Thus the final expression for equation (a) and (b) are equivalent. So,

$$L(h(R)) = h(L(R))$$

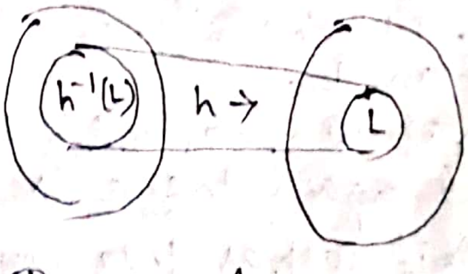
ii) The second case is for concatenation
iii) Third case is for closure.

Inverse Homomorphism

Let h is a homomorphism from some alphabet Σ to strings in another (possibly the same) alphabet T. Let L be a language over alphabet T. Then $h^{-1}(L)$ is the set of strings w in Σ^* such that h(w) is in L.



Forward Homomorphism



Backward or Inverse Homomorphism

$\therefore$ If $h : \Sigma \to T$, $L = L(T)$ then

$$h^{-1}(L) = \{w : w \in \Sigma^* \text{ and } h(w) \in T^*\}$$

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Ex
Let, L be the language of regular expression $(00+1)^*$.

Let h be the homomorphism defined by,
$h(a) = 01$, $h(b) = 10$

Now, we claim that $h^{-1}(L)$ is the language of all regular expression $(ba)^*$, that is all strings of repeating ba pairs.

Now, P.T h(w) is in L iff w is of the form baba---ba :-

Suppose w is n repetitions of ba for some $n \geq 0$. Note that $h(ba) = 1001$, so h(w) is n repetitions of 1001. Now, $1001 \in L$. Therefore any repetition of 1001 is also formed by 1 and 00 segments and is in L. Thus, h(w) is in L.

Now, we'll prove,
if and only if h(w) is in L, then w is of the form baba----ba :-

To prove the above we prove the contrapositive of the statements written below :-

i) if w begins with a, then h(w) begins with 01, It therefore has an isolated 0, and is not in L.

ii) if w ends with b, then h(w) ends with 10, and again there is an isolated 0 in h(w).

iii) if w has two consecutive a's, then h(w) has a substring 0101. Here too, there is an isolated 0 in w.

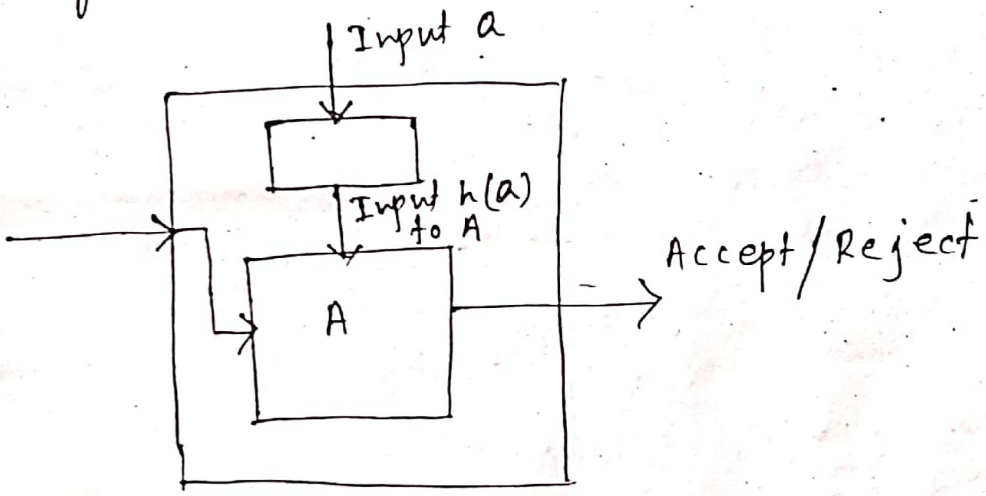
iv) Likewise, if w has two consecutive b's, then h(w) has substring 1010 and has an isolated 0.

Now the logical OR of above four statements is equivalent to the statement " w is not of

the form baba--- ba ...
The statement is contrapositive of the statement we wanted:
"If $h(w)$ is in L, then w is of the form baba---ba".

---

* If h is a homomorphism from alphabet Σ to alphabet T, and L is a regular language over T, then $h^{-1}(L)$ is also a regular language.
— The proof starts with a DFA A for L. We construct from A and h a DFA for $h^{-1}(L)$. This DFA uses the states of A but translates the input symbol according to h before deciding on the next state.



The DFA for $h^{-1}(L)$ applies h to its input, and then simulates the DFA for L

Let, L be L(A), where DFA $A = (Q, T, \delta, q_0, F)$.
Define a DFA B,
$$B = (Q, \Sigma, \gamma, q_0, F) \text{ where}$$
transition function γ is constructed by the rule $\gamma(q, a) = \hat{\delta}(q, h(a))$, That is the transition B makes on input a is the result of the sequence of transitions that A makes on the strings of symbols $h(a)$.
Now, by the method of induction on $|w|$ $\hat{\gamma}(q_0, w) = \hat{\delta}(q_0, h(w))$. Since the accepting state of A and B are same, B accepts w iff A accepts $h(w)$. In other words we can

say B accepts exactly those strings w that are in $h^{-1}(L)$.
