FUNCTIONAL ANALYSIS, PROBLEM SET - I

FOR THE STUDENTS OF INT. M.SC. IN MATHEMATICS, SEMESTER VIII AND
FOR PG STUDENTS OF MATHEMATICS, SEMESTER II
COURSE CODE-MA406/MT 406, YEAR- 2020

1. Normed Linear Spaces

Problem 1.1. Let X be a normed linear space, $a \in X$ and k a non-zero scalar. Prove that the mappings $x \to x+a$ and $x \to kx$ are homeomorphisms of X onto itself

Solution For $a \in X$ and $k(0 \neq) \in K$, let $T_a(x) = x + a,\ x \in X$, and $M_k(x) = kx,\ x \in X$.
Then, $\|T_a(x) - T_a(y)\| = \|x - y\|$
and $\|M_k(x) - M_k(y)\| = |k|\,\|x - y\|$.
These show taht T_a and M_k are uniformly continuous.
Again, $T_{-a}T_a(x) = T_{-a}(x + a) = (x + a) - a = x$,
and $M_{\frac{1}{k}} M_k(x) = M_{\frac{1}{k}}(kx) = x$.
These imply that T_a and M_k have inverses, namely, $T_a^{-1} = T_{-a}$,
and $M_k^{-1} = M_{\frac{1}{k}}$.
These inverses are also of the same type. So they also are uniformly continuous. Hence the result.

Problem 1.2. Let Y be a hyperspace in a normed linear space X, prove that the following are equivalent:

(1) Y is nowhere dense in X.
(2) Y is not dense in X.
(3) Y is closed.

Proof. $(a) \implies (b)$. As Y is nowhere dense in X, then $(\bar{Y})^o$ is empty. Therefore $\bar{Y}$ cannot be equal to X. Hence the proof.
$(b) \implies (c)$. Since$Y \subset \bar{Y} \subset X$ and $\bar{Y}$ is a subspace then $\bar{Y} = Y$ as Y is maximal proper subspace of X and hence Y is closed.
$(c) \implies (a)$ Let if possible $(\bar{Y})^o \neq \phi$. Suppose $a \in (\bar{Y})^o$. Then there exist $r > 0$ such that
$B(a, r) = \{x \in X : \|x - a\| < r\} \subset \bar{Y}$.
Let x be any element of X. If $x \neq 0$, let $y = a + \frac{r}{2\|x\|}x$. Then $y \in B(a, r) \subset \bar{Y}$.
Since $\bar{Y}$ is a subspace we have $x = 2\frac{\|x\|}{r}(y - a) \in \bar{Y}$. This is true even if $x = 0$. So $\bar{Y} = X$. Thus $\bar{Y} \neq Y$ since $Y \neq X$. Which is a contradiction. Hence the proof. □

Problem 1.3. Show that a normed space X is homeomorphic to the open ball $B(0,r)=\{x\in X:\|x\|<r\}, r>0$.

Hint Consider $f:X\to B(0,r)$ defined by $f(x)=\frac{rx}{1+\|x\|}$. For $y\in X$ with $\|y\|<r$ consider $g(y)=\frac{y}{r-\|y\|}$. Show that $f^{-1}=g$ and f is a homeomorphim.

Problem 1.4. Let E_1 and E_2 be subsets of a normed space X, and $E_1+E_2=\{x+y:x\in E_1,y\in E_2\}$. Prove the following:
(1) If E_1 or E_2 is open then E_1+E_2 is open.
(2) If E_1 and E_2 are compact then E_1+E_2 is compact.
(3) If E_1 is compact and E_2 is closed then E_1+E_2 is closed.
(4) E_1+E_2 need not be closed even though E_1 and E_2 are closed.

Problem 1.5. Let $0<p<1$ and $\|x\|_p=(\sum_{j=1}^n |x(j)|^p)^{\frac{1}{p}},\ x\in\mathbb{K}^n$. Show that $\|.\|_p$ is not a norm on $\mathbb{K}^n$ if $n\geq 2$.

Solution The traingle inequlity is not satisfied if $n\geq 2$. For example, let $x=(1,0,0,....,0), y=(0,1,0,....,0)$.
Then $\|x\|_p=1=\|y\|_p$ and $\|x+y\|_p=2^{\frac{1}{p}}$.
Hence $\|x\|_p+\|y\|_p=2<2^{\frac{1}{p}}=\|x+y\|_p$, since $\frac{1}{p}>1$.

Problem 1.6. Let X be the linear space of all polynomials in one variable t with scalar coefficients. For $p\in X$ with $p(t)=a_0+a_1t+....+a_nt^n$, define $\|p\|=sup\{|p(t)|:0\leq\leq 1\},\ \|p\|_1=|a_0|+|a_1|+...+|a_n|$.
Prove that $\|.\|$ and $\|.\|_1$ are norms on X, and $\|p\|\leq\|p\|_1$ for every $p\in X$. Also prove that there is no constant c such that $\|p\|_1\leq c\|p\|$ for all $p\in X$.

Solution First part of the problem is simple. For the last part of the problem, let $p(t)=1-t+t^2-....-t^{2n-1}$.
Now for $0\leq t\leq 1$, we have
$p(t)=(1-t)+t^2(1-t)+...+t^{2n-2}(1-t)\geq 0$,
$p(t)=1-t(1-t)-...-t^{2n-3}(1-t)-t^{2n-1}\leq 1$.
Since $p(0)=1$, we have $\|p\|=1$. Since $\|p\|_1=1+1+....+1=2n$ we have $\|p\|_1>c\|p\|$ if $2n>c$. Thus for any 'c' we can choose a polynomial $p(t)$ with $\|p\|_1>c\|p\|$. Hence the proof.

Problem 1.7. A nonempty set E of a linear space X is said to be balanced if $kx\in E$ whenever $x\in E$ and $k\in\mathbb{K}$ with $|k|\leq 1$. It is said to be absorbing if for every $x\in X$ there exists $r>0$ with $r^{-1}x\in E$. Suppose that E is convex, balanced and absorbing, and also that no nonzero subspace of X is contained in E. For $x\in X$, let $\|x\|=inf\{r>0:r^{-1}x\in E\}$.
Prove that $\|.\|$ is a norm on X, and that $\{x\in X:\|x\|<1\}\subset E\subset\{x\in X:\|x\|\leq 1\}$. Show also that in any normed space X the norm is generated like this by some E.

Solution The first part of the problem is left to the students.
For the second part let X be a normed space, and $E=\{x\in X:\|x\|<1\}$.

It us clear that E is convex, balanced and absorbing.
For each $x \in X$ define $S_x = \{r > 0 : \frac{1}{r}x \in E\}$.
Now $r \in S_x \;\Rightarrow\; r^{-1}x \in E \Rightarrow r^{-1}\|x\| < 1$.
That is $\|x\| < r$ for any $r \in S_x$. This shows that $\|x\| \leq infS_x$.
If possible let $\|x\| < infS_x$, then choose r with $\|x\| < r < infS_x$. Thus $r^{-1}\|x\| < 1$, and so, $r^{-1}\|x\| \in E$; that is, $r \in S_x$. This is not possible as $r < infS_x$ - a contradiction. Hence $\|x\| = infS_x$.

Problem 1.8. Let Y be a closed subspace of a normed space X and $F : X \to X/Y$ be the quotient map defined by $F(x) = x + Y$. Prove that F is continuous mapping and maps open sets of X onto open sets in X/Y.

Problem 1.9. If $x \in L^p(T),\ x \in L^q(T)$ and $p < q$ show that $x \in L^r(T)$ for $p < r < q$.

Proof. We know that e^t is a convex function of t and hence
$e^{\lambda u+(1-\lambda)v} \leq \lambda e^u + (1-\lambda)e^v$ for $0 < \lambda < 1$.
If $t \in T$ and $0 < |x(t)| < \infty$ choose u, v and λ such that
$e^u = |x(t)|^p$, $e^v = |x(t)|^q$ and $r = \lambda p + (1-\lambda)q$.
Thus, $|x(t)|^r = \lambda\,|x(t)|^p + (1-\lambda)\,|x(t)|^q$.
Therefore, integrating both side of above we have
$\int |x|^r\,d\mu \leq \lambda \int |x|^p\,d\mu + (1-\lambda)\int |x|^q\,d\mu < \infty$.
Hence $x \in L^r(T)$. □

Problem 1.10. Prove that if $x \in l^r$ for some $r \geq 1$ then $x \in l^p$ for all $p > r$. Prove also that $lim_{p\to\infty}\|x\|_p = \|x\|_\infty$, for any $x \in l^r$.

Problem 1.11. Show that $c_{00} \subset l^p$ for $1 \leq p \leq \infty$. Is it true that $c_0 \subset l^p$?

Solution First part is clear, for the second part take $x(n) = n^{-1/p}$. Then $x \in c_0$ but $x \notin l^p$ as $\sum |x(n)|^p = \sum \frac{1}{n} = \infty$. So, $c_0 \subset l^p$ is not true for $1 \leq p < \infty$. But as we know that every convergent sequence is bounded hence it is true for $p = \infty$.

Problem 1.12. Prove that l^∞ is a Banach space. Show that c and c_0 are closed subspaces of l^∞ and hence Banach spaces. Find the dimension of the quotient space c/c_0.

Problem 1.13. Show that c_{00} is not a closed subspace of $l^\infty$and l^2.

Solution It is clear that c_{00} is a subspace of l^2 as well as l^∞. We shall show that it is not closed in these spaces.
Let

$$x_n(j) = \begin{cases} \frac{1}{j}, & 1 \leq j \leq n \\ 0, & j > n \end{cases}$$

and $x(j) = \frac{1}{j}$ for all $j \geq 1$.
Then $x_n \in c_{00}$ for all n, and $x \in l^2 \subset l^\infty$.
Now $x - x_n = (0, ...0, \frac{1}{n+1}, \frac{1}{n+2}, ...)$.

So, $\|x - x_n\|_\infty = \frac{1}{n+1} \to 0$ as $n \to \infty$,
and $\|x - x_n\|_2^2 = \sum_{j=n+1}^{\infty} \frac{1}{j^2} \to 0$ as $n \to \infty$.
Hence $x_n \to x$ in l^∞ and l^2. Therefore, x is in the closure of c_{00} in l^∞ and in l^2. As $x \notin c_{00}$, hence c_{00} is not closed in l^∞ and l^2.

Problem 1.14. Show that there exist two vectors x and y in the space l^∞ such that they are linearly independent, $\|x\| = \|y\| = 1$ and $\|x + y\| = 2$.

Problem 1.15. Prove that if a normed linear space contains linearly independent vectors x and y such that $\|x\| = \|y\| = 1$ and $\|x + y\| = \|x\| + \|y\|$, then there exists a line segment contained in the unit sphere of X.

2. Continuity of linear maps

Problem 2.1. Let f be a non-zero linear functional on a linear space X. Show that $Z(f)$ (zero space of f) is a hyperspace in X. Also show that any $x \in X$ has a unique representation as $x = kx_0 + y$ with $k \in \mathbb{K}$, $y \in Z(f)$ and for some x_0 so that $f(x_0) \neq 0$.

Problem 2.2. Show that for each hyperspace Z of a linear space X there is a linear functional f on X such that $Z = Z(f)$.

Problem 2.3. Let f_1 and f_2 be two linear functional on a linear space X. Show that if they have the same zero space then there is a nonzero scalar k such that $f_2(x) = kf_1(x)$ for all $x \in X$.

Hint. If $Z(f_1) = X$ nothing to show. Let $f(x_0) \neq 0$. Choose $k = \frac{f_2(x_0)}{f_1(x_0)}$. For any $x \in X$ let $\alpha = \frac{f_1(x)}{f_1(x_0)}$. Now show that $x - \alpha x_0 \in Z(f_1) = Z(f_2)$. Hence $f_2(x) = \alpha f_2(x_0) = \alpha k f_1(x_0) = kf_1(x)$.

Problem 2.4. Let X be a normed space and f be a nonzero bunded linear functional on X. Show that
$E = \{x \in X : f(x) = \|f\|\}$ is a nonempty closed convex subset of X. Also show that $\inf\{\|x\| : x \in E\} = 1$.

Hint. By the given condition, there exist some $x_0 \in X$ such that $f(x_0) \neq 0$. Choose $k = \frac{\|f\|}{f(x_0)}$. show that $kx_0 \in E$; thus E is nonempty.
Let $x \in \bar{E}$, then there is (x_n) in E such that $x_n \to x$. Now applying continuity of f show that $f(x) = \|f\|$; and hence $x \in E$.
For $x, y \in E,\ t \in [0, 1]\ f(tx + (1 - t)y) = \|f\|$; hence E is convex.

Problem 2.5. Show that $f : (c_{00}, \|.\|_\infty) \to \mathbb{K}$ defined by $f(x) = \sum_{n=1}^{\infty} x(n)$ is not continuous.

Hint. Consider $x_n \in c_{00}$ defined by $x_n(j) = 1\ if\ 1 \leq j \leq n$ otherwise 0. Then $\|x_n\|_\infty = 1$ and $f(x_n) = n$. If f were continuous then $|f(x_n)| \leq c\|x_n\|_\infty = c$(for some $c > 0$), imply $n \leq c$ for all $n \in \mathbb{N}$. Which is a contradiction.

Problem 2.6. Let $f(x) = lim_{n\to\infty} x(n)$, and $g(x) = \sum \frac{x(n)}{n^2}$. Show that f and g are bounded linear functionals on c and l^∞ respectively. Find their norms.

Solution As c is a linear space then $(kx + y)(n) = kx(n) + y(n)$ for $x, y \in c$, $k \in \mathbb{K}$ and for all $n \in \mathbb{N}$. Taking $n \to \infty$ we have,
$f(kx + y) = kf(x) + f(y)$. Hence f is linear.
for $x \in c$, we have $|x(n)| \leq \|x\|_\infty$ for all n.
By the given definition of f, we have $|f(x)| \leq \|x\|_\infty$. This shows that f is bounded and $\|f\| \leq 1$. Again if $x(n) = 1, \forall n \in \mathbb{N}$, then $x \in c$ and $\leq \|x\|_\infty = 1$. Therefore,
$1 = f(x) \leq \|f\| \leq \|x\|_\infty = \|f\|, \Rightarrow \|f\| = 1$.
For g it is easy to linearity. For norm, if $x \in l^\infty$ then $x(n) \leq \|x\|_\infty$ for all n; hence
$|g(x)| \leq \|x\|_\infty \sum_{n=1}^\infty \frac{1}{n^2} = \|x\|_\infty \frac{\pi^2}{6}$.
This shows that $\|g\| \leq \frac{\pi^2}{6}$. Again, if $x(n) = 1$ for all n, then $x \in l^\infty,\ g(x) = \frac{\pi^2}{6}$, and $\|x\|_\infty = 1$. Thus $\frac{\pi^2}{6} = g(x) \leq \|g\| \|x\|_\infty = \|g\|$.
It follows that $\|g\| = \frac{\pi^2}{6}$.

Problem 2.7. Find the norm of the linear functional f deined on $C[-1, 1]$ by
$f(x) = \int_{-1}^0 |x(t)|\, dt - \int_0^1 |x(t)|\, dt,\ x \in C[-1, 1]$.

Hint. It is easy to show that $\|f\| \leq 2$.
For $n = 1, 2, ...,$ let

$$x_n(t) = \begin{cases} 1, & -1 \leq t \leq -1/n \\ -nt, & -1/n < t < 1/n \\ -1, & 1/n \leq t \leq 1. \end{cases}$$

Then, $x_n \in C[-1, 1],\ \|x\|_\infty = 1$, and $f(x_n) = 2 - 1/n$.
Therefore, $2 - 1/n \leq \|f\|,\ \forall n$. Hence $\|f\| = 2$.

Problem 2.8. Define $S,\ T : C[0, 1] \to C[0, 1]$ by
$(Sx)(s) = s \int_0^1 x(t)dt$, $(Tx)(s) = sx(s)$ for $0 \leq s \leq 1$.
Do S and T commute? Find $\|S\|,\ \|T\|,\ \|ST\|$ and $\|TS\|$.

Answer $ST \neq TS$, $\|S\| = \|T\| = \|TS\| = 1$ and $\|ST\| = \frac{1}{2}$.

Problem 2.9. Consider $\mathbb{R}^2$ with Euclidean norm and $A \in BL(\mathbb{R}^2)$ be represented by the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Show that $\|A\| = \frac{1}{2}(\sqrt{\alpha} + \sqrt{\beta})$, where $\alpha = a^2 + b^2 + c^2 + d^2 + 2(ad - bc)$ and $\alpha = a^2 + b^2 + c^2 + d^2 - 2(ad - bc)$.

Hint Consider $x = (cos\theta, sin\theta) \in \mathbb{R}^2$.
Then $\|Ax\|^2 = \frac{a^2+b^2+c^2+d^2}{2} + \frac{a^2+c^2-b^2-d^2}{2} cos(2\theta) + (ab + cd) sin(2\theta)$
$= \frac{\alpha+\beta}{4} + rcos(2\theta - \phi)$
where $\alpha,\ \beta$ are as above and $rcos\phi = \frac{a^2+c^2-b^2-d^2}{2}$, $rsin\phi = ad + bc$.

So, $4r^2 = \alpha\beta$.
Hence $\|Ax\|^2 = \frac{\alpha+\beta}{4} + \frac{\sqrt{\alpha\beta}}{2} cos(2\theta - \phi) \leq (\frac{\sqrt{\alpha}+\sqrt{\beta}}{2})^2$,
and equality holds for $\theta = \frac{1}{2}\phi$.

References

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