Practice Paper I, REAL ANALYSIS I

1. What do you mean by a denumerable set? Cite an example. Prove that the union of a denumerable collection of denumerable sets is also denumerable. Use this to prove that the set $\mathbb{Q}$ of all rational numbers is denumerable.

2. What do you mean by the terms infimum and supremum? State L.U.B. axiom and use it to prove G.L.B. property.

3. Let A be a bounded set of real numbers with $M = \sup A$ and $m = \inf A$. We define another set B by

   $$B = \{|x - y| : x, y \in A\}$$

   Find $\sup B$ and $\inf B$.

   What would happen if the modulus sign used in the definition of the set B is removed?

4. Prove that the set of all rational numbers is not complete.

   [Hints: Show that for any set like

   $$S = \{x \in \mathbb{Q} : x > 0, x^2 < 2\},$$

   the set S is bounded above and is non-empty but $\sup S \notin S$, and claim accordingly.]

5. State and prove the Archimedean property for real numbers. Use it to prove that the greatest integer function is well-defined.

   [Hints: For any $x \in \mathbb{R}$, prove that there exists unique $n \in \mathbb{Z}$ satisfying

   $$n \leq x < (n + 1).]$$

6. Let $x, y \in \mathbb{R}$ with $x < y$. Prove that the open interval (x, y) contains at least one rational number and at least one irrational number.

7. What do you mean by “Arithmetic Continuum" and “Linear Continuum"? State Cantor-Dedekind Axiom.

8. Define an interval. Are the empty set $\emptyset$ and any singleton set intervals? Justify your answer. If an interval contains at least two distinct points can we say that it contains uncountably many points?