Frequency Dispersion in Materials

Dispersion is the phenomenon in which the phase velocity of a wave depends on its frequency. Media having such a property are termed dispersive media. The dispersion relation of an EM wave in a dispersive medium is expressed as

$$k = n(\omega)\frac{\omega}{c}$$

Normal dispersion:

When a white light is passed through a prism color separation is observed. Light being electro-magnetic oscillations, the different colors have different wavelengths and different frequencies in vacuum for all of them move with the same speed c. Our eyes sense different wavelengths of visible spectrum with different colors. The separation happens because the different wavelengths have different refractive indices. Whenever light enters a dielectric medium this separation happens and the phenomenon is known as dispersion. In simple term it the variation of refractive index with the wavelength. In other words, the variation of the frequency with the wavelength in a medium is dispersion. Cauchy studied dispersion and gave a formula which described the dispersion in the visible range quite well. The following formula is known as Cauchy's dispersion formula,

$$n(\lambda) = A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4}$$

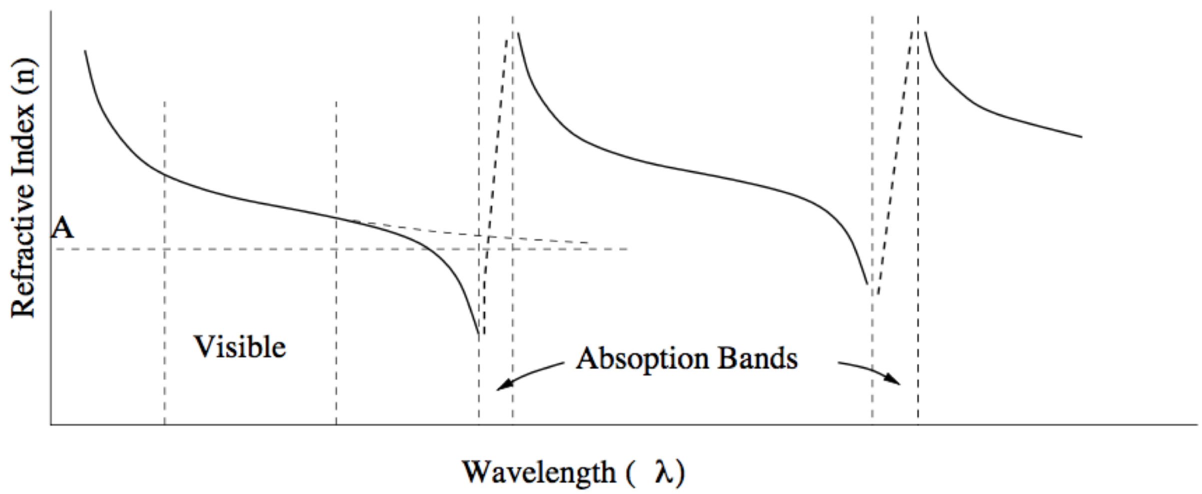
where A, B and C are constants which depend on the medium. Experimentally the constants can be determined be measuring the the refractive index for three wavelengths. In usual condition the first two terms would suffice to give an accurate value of n. The derivative of the refractive index is given by

$$\frac{dn}{d\lambda} = -\frac{C}{\lambda^3}$$

to a good accuracy. Since A and B both are positive the refractive index decreases increasing the wavelength.

Anomalous dispersion:

For material transparent to visible region Cauchy's formula works very well but if one further increases the wavelength say to the infra-red, one finds the refractive index suddenly decreases very fast and does not obey the Cauchy's law. One now approaches the absorption region. Further increasing the wavelength once again refractive index becomes large. Again the behavior is quite similar to the visible region for the increase in wavelength. If the range is increased further one again observes another absorption band as shown in the figure 1 below. The pattern may repeat further as shown, giving many absorption bands. This dispersion is known as anomalous dispersion.



The first theory of it came from Sellmeier who assumed that all elastically bound particles in the medium oscillate with a natural frequency ω_0 which correspond to a wavelength λ_0 in the vacuum. Sellmeir, formula gave,

$$n^2 = 1 + \frac{A\lambda^2}{\lambda^2 - \lambda_0^2}$$

where A is a constant. If one is away from λ_0 it can be expanded in powers of λ_0/λ and one would get a formula of the Cauchy type.

Drude-Lorentz harmonic oscillator model

All ordinary matters are composed of electrons and nuclei. The bound electrons can be treated as harmonic oscillators. For generality we make it a damped harmonic oscillator. When an EM wave is present, the oscillator is driven by the electric field of the wave. The response of the medium is obtained by adding up the motions of the electrons. The equation of motion for an electron of charge *–e* and acted on by an electric field $\boldsymbol{E}(\boldsymbol{x}, t)$ is

$$\frac{d^2\boldsymbol{x}}{dt^2} + \gamma\frac{d\boldsymbol{x}}{dt} + \omega_0^2\boldsymbol{x} = -\frac{e}{m}\boldsymbol{E}(\boldsymbol{x}, t)$$

where the damping constant has the dimensions of frequency. The amplitude of oscillation is small compared to the spatial variation of the field. Assuming that the field varies harmonically in time with frequency as , the dipole moment contributed by one electron is

$$\boldsymbol{p} = -e\boldsymbol{x} = \frac{e^2}{m}\frac{\boldsymbol{E}}{\omega_0^2 - \omega^2 - i\gamma\omega}$$

If there are *N* molecules per unit volume with *Z* electrons per molecule, and there are f_j electrons per molecule with binding frequency ω_j and damping constant γ_j, then the dielectric constant is given by

$$\varepsilon_r = \frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \chi_e(\omega) = 1 + \frac{Ne^2}{\varepsilon_0 m}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - j\gamma_j\omega}$$

Resonant absorption and anomalous dispersion

In a dispersive medium (nonmagnetic), plane waves are expressed as

$$\boldsymbol{E}(\boldsymbol{z}, t) = \boldsymbol{E}_0 e^{i(kz-\omega t)}$$

with the complex wave number

$$k(\omega) = \sqrt{\varepsilon\mu_0}\omega = n(\omega)\frac{\omega}{c}$$

Writing in terms of its real and imaginary parts,

$$k = \alpha + i\beta$$

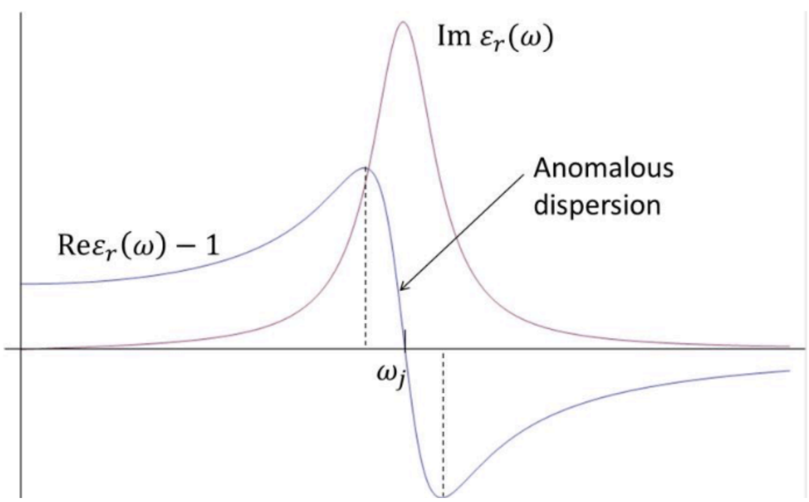
with the attenuation constant or absorption coefficient β

$$\boldsymbol{E}(\boldsymbol{z},t) = \boldsymbol{E}_0 e^{-\beta z} e^{i(\alpha z - \omega t)}$$

Evidently the wave is exponentially attenuated because the damping absorbs energy. The intensity of the wave falls off as $e^{-2\beta z}$. The relation between (α, β) and ε_r

$$\alpha^2 - \beta^2 = \frac{\omega^2}{c^2} Re\varepsilon_r$$

$$\alpha\beta = \frac{\omega^2}{c^2} Im\varepsilon_r$$



The general features of the real and imaginary parts of ε_r around a resonant frequency are shown in figure. Most of the time $Re\varepsilon_r$ (or the index of refraction with small α) rises gradually with increasing frequency (normal dispersion). However, in the immediate neighborhood of a resonance $Re\varepsilon_r$ drops sharply. Because this behavior is atypical, it is called anomalous dispersion. Notice that the region of anomalous dispersion coincides with the region of maximum absorption