MA310: Linear Programming Problem

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1 General Form

General form of an LPP is:

$$\left.\begin{array}{rl} \text{Maximize/ Minimize} & z = f(x_1, x_2, \ldots, x_l) \\ \text{subject to} & g_j(x_1, x_2, \ldots, x_l) \begin{array}{c}\le\\=\\\ge\end{array} b_j,\ j = 1, 2, \ldots, m \\ & x_i \ge 0,\ i = 1, 2, \ldots, l. \end{array}\right\} \tag{1}$$

where f and g_j, $j = 1, 2, \ldots, m$ are linear functions of the *decision variables* $x_1, x_2, \ldots, x_l$. The function f is called the *objective function* and $g_j \begin{array}{c}\le\\=\\\ge\end{array} b_j$, $j = 1, 2, \ldots, m$ where either one of the relations '$\le$', '$=$', '$\ge$' holds, are called *constraints*.

It may be noted that if the problem is to minimize the objective function f, it can be converted to a maximization problem with the objective function $-f$. It may also be observed that each of the constraints can be written as an equality —- by adding some non-negative quantity if it is of '$\le$'-type, and by subtracting some non-negative quantity if it is of '$\ge$'-type. The non-negative quantity x_k added to g_k to make $g_k \le b_k$ an equality, is called a *slack variable* and the non-negative quantity x_r subtracted from g_r to make $g_r \ge b_r$ an equality is called a *surplus variable*. Thus by considering the given problem as a maximization problem and introducing slack and surplus variables, an LPP can always be written in the *standard form* as follows:

$$\left.\begin{array}{rc} \text{Maximize} & z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{subject to} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \\ & x_i \ge 0,\ i = 1, 2, \ldots, n. \end{array}\right\} \tag{2}$$

Using vector and matrix notations, an LPP in standard form can be written as:

$$\left.\begin{array}{rc} \text{Maximize} & z = \boldsymbol{cx} \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \ge 0 \end{array}\right\} \tag{3}$$

where $\boldsymbol{c} = (c_1, c_2, \ldots, c_n)$ is called the cost vector, $A = (a_{ij})_{m\times n}$ is the coefficient matrix, $\boldsymbol{b} = (b_1, b_2, \ldots, b_m)^T$ is the requirement vector and $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$ is the decision vector.

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2 Basic Feasible Solution

We suppose that all the rows of the augmented matrix $[A : \boldsymbol{b}]$ in the standard LPP (3) are independent. Because if it is not so, all the redundant rows (i.e., the rows which can be written as linear combination of the others) can be dropped to make so. Now we suppose that the ith column of the matrx A is $\boldsymbol{a}_i$, i.e. $\boldsymbol{a}_i = (a_{1i}, a_{2i}, \dots, a_{mi})^T$. Then $A\boldsymbol{x} = \boldsymbol{b}$ is same as writing

$$\boldsymbol{a}_1 x_1 + \boldsymbol{a}_2 x_2 + \cdots + \boldsymbol{a}_n x_n = \boldsymbol{b}. \tag{4}$$

A vector $\boldsymbol{x} \in \mathbb{R}^n$ is said to be a *feasible solution* of the LPP (3) if $A\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \geq 0$. Clearly, a feasible solution of an LPP does not depend on the objective function, rather it depends on the constraints only. Now, if the system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$ is inconsistent (i.e., rank$([A : \boldsymbol{b}]) \neq$ rank(A)), then the LPP (3) has no feasible solution. Also if $A\boldsymbol{x} = \boldsymbol{b}$ is consistent and has a unique solution $\boldsymbol{x} \geq 0$, then that is the optimal solution of (3). In that case, n must be equal to m. Thus, we consider the general case: $A\boldsymbol{x} = \boldsymbol{b}$ is consistent and $m < n$, i.e., rank$([A : \boldsymbol{b}]) =$ rank$(A) = m < n$. Then $A\boldsymbol{x} = \boldsymbol{b}$ has infinite number of solutions, some of which are possibly feasible, i.e., $\boldsymbol{x} \geq 0$.

If m columns of A, say, $\boldsymbol{a}'_1, \boldsymbol{a}'_2, \dots, \boldsymbol{a}'_m$ are independent, then they form a basis of the column space of A. The matrix $B = \left[\boldsymbol{a}'_1, \boldsymbol{a}'_2, \dots, \boldsymbol{a}'_m\right]$ is called a basis matrix of the LPP (3). Now, after possible rearrangements of the columns, the matrix A can be partitioned as $\left[B : NB\right]$. We partition the vector $\boldsymbol{x}$ with corresponding components as $(\boldsymbol{x}_B, \boldsymbol{x}_{NB})^T$. If the $n-m$ variables corresponding to the columns of A in NB are taken as zero, then the equation (4) reduces to $B\boldsymbol{x}_B = \boldsymbol{b}$ and since B is invertible, this is uniquely solvable as $\boldsymbol{x}_B = B^{-1}\boldsymbol{b}$. Such a solution $\boldsymbol{x} = (\boldsymbol{x}_B, \overbrace{0, 0, \dots, 0}^{n-m})^T$ is called a *basic solution* of (3). Variables in $\boldsymbol{x}_B$ are called *basic variables* and the variables in $\boldsymbol{x}_{NB}$ are called *non-basic variables.*

Definition 1 *A solution obtained by setting exactly $n-m$ variables of an LPP with n variables and m independent equality constraints, equal to zero, provided the determinant formed by the columns associated with the remaining m variable is non-zero, is called a basic solution of the LPP.*

Definition 2 *In an LPP, a feasible solution which is also a basic solution is called a basic feasible solution (BFS) of the LPP.*

Definition 3 *If any of the basic variable takes the value zero in a basic solution of an LPP, that basic solution is termed as a degenerate basic solution otherwise it is called a non degenerate basic solution.*

Example 1 *Consider an LPP with the following constraints:*

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 4 \\ 2x_1 + x_2 + 5x_3 &= 5. \end{aligned}$$

Find all the basic feasible solutions of the LPP.

Solution. Here, $\boldsymbol{a}_1 = \begin{pmatrix}1\\2\end{pmatrix}, \boldsymbol{a}_2 = \begin{pmatrix}2\\1\end{pmatrix}, \boldsymbol{a}_3 = \begin{pmatrix}1\\5\end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix}4\\5\end{pmatrix}$. If we choose, $B = \left(\boldsymbol{a}_2, \boldsymbol{a}_3\right) = \begin{pmatrix}2 & 1\\1 & 5\end{pmatrix}$, then $\det(B) \neq 0$. So, B can be choosen as a basis matrix with x_2, x_3 as basic variables and x_1 as non-basic variable. Now, $\begin{pmatrix}x_2\\x_3\end{pmatrix} = B^{-1}\boldsymbol{b} = \begin{pmatrix}5/3\\2/3\end{pmatrix}$ and so, $\begin{pmatrix}0\\5/3\\2/3\end{pmatrix}$ is a basic solution.

Similarly, we can find the other basic solutions $\begin{pmatrix}5\\0\\-1\end{pmatrix}$ and $\begin{pmatrix}2\\1\\0\end{pmatrix}$. Among these three basic solutions, $\begin{pmatrix}5\\0\\-1\end{pmatrix}$ is not feasible. Hence the basic feasible solutions are $\begin{pmatrix}0\\5/3\\2/3\end{pmatrix}$ and $\begin{pmatrix}2\\1\\0\end{pmatrix}$.

Remark 1 *All the basic solutions in the above example are non-degenerate since no basic variable has the value zero.*

Example 2 *Consider an LPP with the following constraints:*

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 4 \\ x_1 + x_2 + 2x_3 &= 2. \end{aligned}$$

Find all the basic feasible solutions of the LPP.

Solution. Here, $\boldsymbol{a}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \boldsymbol{a}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \boldsymbol{a}_3 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$. If we choose, $B = (\boldsymbol{a}_2, \boldsymbol{a}_3) = \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}$, then $\det(B) \neq 0$. So, B can be choosen as a basis matrix with x_2, x_3 as basic variables and x_1 as non-basic variable. Again, $\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = B^{-1}\boldsymbol{b} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and so, $\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$ is a basic solution.

But if we choose $B = (\boldsymbol{a}_1, \boldsymbol{a}_3) = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$, then $\det(B) = 0$ and so setting $x_2 = 0$ does not yeald a basic solution. Choosing $B = (\boldsymbol{a}_1, \boldsymbol{a}_2) = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ and setting $x_3 = 0$, we get the other basic solution also $\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$.

Thus, $\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$ is the only BFS of the given LPP.

Remark 2 *The basic solutions in the above example is degenerate since a basic variable has also the value zero.*

3 Fundamental Theorem

Theorem 1 *Given the LPP*

$$\begin{aligned} \textit{Maximize} \quad & z = \boldsymbol{c}\boldsymbol{x} \\ \textit{subject to} \quad & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{aligned}$$

where A is an $m \times n$ matrix, $m < n$ and rank$(A) = m$.

1. *If there is a feasible solution, then there is a BFS.*
2. *If there is an optimal solution, then there is a basic optimal solution.*

Proof. Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ be a feasible solution of given system of equations $A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq 0$.

Suppose, out of these n components of $\boldsymbol{x}$, k are non-zero and rest $(n-k)$ are zero. Without loss of generality, we assume that first k components of $\boldsymbol{x}$ are non-zero. Thus, we have $\boldsymbol{x} = (x_1, x_2, \ldots, x_k, \overbrace{0, 0, \ldots, 0}^{n-k})^T$.

Since $\boldsymbol{x}$ is a feasible solution, we have

$$A\boldsymbol{x} = \boldsymbol{b} \Rightarrow \sum_{j=1}^{k} \boldsymbol{a}_j x_j = \boldsymbol{b} \tag{5}$$

and $\boldsymbol{x} \geq 0 \Rightarrow x_j > 0, \forall j = 1, 2, \ldots, k$. Then there are two possibilities.

(i) $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_k\}$ are linearly independent. Then we must have $k \leq m$. If $k = m$, the corresponding solution is basic and the proof is complete. If $k < m$ then, since A has rank m, we choose $m-k$ vectors from the remaining $n-k$ columns of A so that the resulting set of m column vectors is linearly independent. Assigning the value zero to the corresponding $m-k$ variables yields a (degenerate) basic feasible solution.

(ii) $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_k\}$ are not linearly independent. Then there exist scalars $\lambda_1, \lambda_2 \ldots, \lambda_k$ not all zero such that

$$\sum_{j=1}^{k} \lambda_j \boldsymbol{a}_j = 0 \tag{6}$$

Suppose $\lambda_r \neq 0$. Without loss of generality, we take $\lambda_r > 0$. Because if it is not so, then equation (3) can be multiplied by -1 on both sides to make $\lambda_r > 0$. Now $\boldsymbol{a}_r = -\sum_{j=1, j\neq r}^{k} \dfrac{\lambda_j}{\lambda_r} \boldsymbol{a}_j$. Then from (5), we get

$$\sum_{j=1, j\neq r}^{k} \left(x_j - \frac{\lambda_j}{\lambda_r} x_r \right) \boldsymbol{a}_j = \boldsymbol{b}$$
$$\Rightarrow \sum_{j=1}^{k} \hat{x}_j \boldsymbol{a}_j = \boldsymbol{b}$$

where, $\hat{x}_j = x_j - \frac{\lambda_j}{\lambda_r} x_r \ \forall j = 1, 2, \ldots, k, j \neq r$ and $\hat{x}_r = 0$.

Thus $\hat{\boldsymbol{x}} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_k, \overbrace{0, 0, \ldots, 0}^{n-k})^T$ is a feasible solution of $A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq 0$ if $\hat{\boldsymbol{x}} \geq 0$, i.e., if $x_j - \frac{\lambda_j}{\lambda_r} x_r \geq 0 \ \forall j = 1, 2, \ldots, k, j \neq r$.

Now if $\lambda_j \leq 0$, then $\hat{x}_j$ is automatically non-negative since $x_j > 0, x_r > 0$ and $\lambda_r > 0$. So, our concern are those $\hat{x}_j$'s for which $\lambda_j > 0$. Now, if we choose r in such a manner that

$$\frac{x_r}{\lambda_r} = \min \left\{ \frac{x_j}{\lambda_j} : \lambda_j > 0 \right\} \tag{7}$$

then $x_j - \frac{\lambda_j}{\lambda_r} x_r \geq 0 \ \forall j = 1, 2, \ldots, k, j \neq r$, i.e., $\hat{x}_j \geq 0 \ \forall j = 1, 2, \ldots, k, j \neq r$ and $\hat{x}_r = 0$.

Thus $\hat{\boldsymbol{x}}$ is a feasible solution of $A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq 0$ with at most $k-1$ non-zero components. If $k-1 = m$, then $\hat{\boldsymbol{x}}$ is a BFS. Otherwise, we repeat the above procedure at most $k-1$ times, so that a feasible solution with at most m non-zero components is yielded and that solution will obvoiusly be a BFS.

This completes the proof of the first part of the theorem.

Now assume that $\boldsymbol{x}^*$ is an optimal solution. There is no guarantee that this optimal solution is unique. Some of these solutions may have more positive components than others. Without loss of generality, we assume that $\boldsymbol{x}^*$ has a minimal number of positive components. If $\boldsymbol{x}^* = 0$ then $\boldsymbol{x}^*$ is basic and the optimal cost is zero. If $\boldsymbol{x}^* \neq 0$ and if we choose $\boldsymbol{x}^* = (x_1^*, x_2^*, \ldots, x_k^*, \overbrace{0, 0, \ldots, 0}^{n-k})^T$ then there are two cases as before. The proof of the first case in which the corresponding columns are linearly independent is exactly same as in the previous proof of this case.

In the second case, we have scalars $\lambda_1, \lambda_2 \ldots, \lambda_k$ not all zero such that $\sum_{j=1}^{k} \lambda_j \boldsymbol{a}_j = 0$ as before. Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_k, \overbrace{0, 0, \ldots, 0}^{n-k})^T$. Then for any real number α, $\boldsymbol{x}^* - \alpha\boldsymbol{\lambda}$ is a solution of $A\boldsymbol{x} = 0$ since $A\boldsymbol{x}^* = \boldsymbol{b}$ and $A\boldsymbol{\lambda} = 0$. We shall show that $\boldsymbol{x}^* - \alpha\boldsymbol{\lambda}$ is also an optimal solution for suitable choice of α. We note that $\boldsymbol{c}(\boldsymbol{x}^* - \alpha\boldsymbol{\lambda}) = \boldsymbol{c}\boldsymbol{x}^* - \alpha\boldsymbol{c}\boldsymbol{\lambda}$.

Now, for sufficiently small $|\alpha|$, the vector $\boldsymbol{x}^* - \alpha\boldsymbol{\lambda}$ is a feasible solution for positive or negative values of α. Hence, we conclude that $\boldsymbol{c}\boldsymbol{\lambda} = 0$, because otherwise we could determine a small α of the proper sign, to make $\boldsymbol{c}(\boldsymbol{x}^* - \alpha\boldsymbol{\lambda}) > \boldsymbol{c}\boldsymbol{x}^*$, which would violate the assumption of the optimality of $\boldsymbol{x}^*$. Having established that the new feasible solution with fewer positive components is also optimal, the remainder of the proof may be completed by choosing $\alpha = \dfrac{x_r}{\lambda_r} = \min\left\{\dfrac{x_j}{\lambda_j} : \lambda_j > 0\right\}$ as in case (ii) above. □

Example 3 *Reduce the feasible solution* $x_1 = 2, x_2 = 3, x_3 = 1$ *to a basic feasible solution of the LPP:*

$$\begin{aligned} \max z = x_1 + 2x_2 + 4x_3 & \\ \textit{subject to}\; 2x_1 + x_2 + 4x_3 &= 11 \\ 3x_1 + x_2 + 5x_3 &= 14 \\ x_1, x_2, x_3 \geq 0. & \end{aligned}$$

Solution. Here the columns of A associated with $x_i > 0, i = 1,2,3$ are $\boldsymbol{a}_1 = \begin{pmatrix}2\\3\end{pmatrix}, \boldsymbol{a}_2 = \begin{pmatrix}1\\1\end{pmatrix}$ and $\boldsymbol{a}_3 = \begin{pmatrix}4\\5\end{pmatrix}$ which forms a linearly dependent set since three vectors of $\mathbb{R}^2$ can not be independent. Moreover, $1\begin{pmatrix}2\\3\end{pmatrix} + 2\begin{pmatrix}1\\1\end{pmatrix} - 1\begin{pmatrix}4\\5\end{pmatrix} = 0$, i.e., $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = -1$ (values of $\lambda_1, \lambda_2, \lambda_3$ can be obtained by cross multiplication between the equations obtained from $\lambda_1\boldsymbol{a}_1 + \lambda_2\boldsymbol{a}_2 + \lambda_3\boldsymbol{a}_3 = 0$.)

Choose r such that

$$\begin{aligned} \frac{x_r}{\lambda_r} &= \min\left\{\frac{x_j}{\lambda_j} : \lambda_j > 0\right\} \\ &= \min\left\{\frac{x_1}{\lambda_1}, \frac{x_2}{\lambda_2}\right\} \\ &= \min\left\{2, \frac{3}{2}\right\} \\ &= \frac{3}{2} \\ &= \frac{x_2}{\lambda_2} \end{aligned}$$

i.e., $r = 2$.

Hence, $\hat{\boldsymbol{x}} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)^T$ where $\hat{x}_1 = x_1 - \frac{\lambda_1}{\lambda_2}x_2 = \dfrac{1}{2}, \hat{x}_2 = 0$ and $\hat{x}_3 = x_3 - \frac{\lambda_3}{\lambda_2}x_2 = \dfrac{5}{2}$ is another feasible solution containting two non-zero components. Hence, $\hat{\boldsymbol{x}} = \left(\dfrac{1}{2}, 0, \dfrac{5}{2}\right)^T$ is a BFS. Note that the columns $\boldsymbol{a}_1$ and $\boldsymbol{a}_3$ of A corresponding to the non-zero components of $\hat{\boldsymbol{x}}$ are linearly independent.

4 Simplex Method

As stated in the fundamental theorem of LPP, *if an LPP has an optimal solution, then that is either a BFS or there is another optimal solution which is a BFS*. In other words, optimal solution of an LPP, if exists, can be found only at the BFSs. Simplex method relies on this theorem It starts with an initial BFS and go on searching a new BFS improving the objective function value until some optimality condition is reached, i.e., no further improvement is possible.

4.1 Improvement of a BFS and optimality condition

Suppose $(\boldsymbol{x}_B, \overbrace{0, 0, \ldots, 0}^{n-m})^T$ is a BFS of LPP (3), where A is an $m \times n$ matrix with rank m, $\boldsymbol{b}$ is $m \times 1$ and $\boldsymbol{c}$ is $n \times 1$ vector and $B = [\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \ldots, \boldsymbol{\beta}_m]$ be the associated basis matrix. Now suppose there exists a column $\boldsymbol{a}_j$ in A which is not in B. Then there exist scalars $y_{ij}, i = 1,2,\ldots,m$ such that $\boldsymbol{a}_j = \sum_{i=1}^{m} y_{ij}\boldsymbol{\beta}_i$. Let $\boldsymbol{y}_j = (y_{1j}, y_{2j},\ldots, y_{mj})^T$. Then $\boldsymbol{a}_j = \sum_{i=1}^{m} y_{ij}\boldsymbol{\beta}_i = B\boldsymbol{y}_j \Rightarrow \boldsymbol{y}_j = B^{-1}\boldsymbol{a}_j$.

For a moment assume that there is some scalar $y_{rj} > 0$. Then

$$\boldsymbol{\beta}_r = -\sum_{i=1,i\neq r}^{m} \frac{y_{ij}}{y_{rj}}\boldsymbol{\beta}_i + \frac{1}{y_{rj}}\boldsymbol{a}_j.$$

Since $(\boldsymbol{x}_B, \overbrace{0,0,\ldots,0}^{n-m})^T$ is a BFS, so

$$\begin{aligned}
& B\boldsymbol{x}_B = b \\
\Rightarrow\quad & \sum_{i=1}^{m} x_{B_i}\boldsymbol{\beta}_i = b \\
\Rightarrow\quad & \sum_{i=1,i\neq r}^{m} x_{B_i}\boldsymbol{\beta}_i + x_{B_r}\left(-\sum_{i=1,i\neq r}^{m}\frac{y_{ij}}{y_{rj}}\boldsymbol{\beta}_i + \frac{1}{y_{rj}}\boldsymbol{a}_j\right) = b \\
\Rightarrow\quad & \sum_{i=1,i\neq r}^{m} \hat{x}_{B_i}\boldsymbol{\beta}_i + \hat{x}_{B_r}\boldsymbol{a}_j = b
\end{aligned}$$

where, $\hat{x}_{B_i} = x_{B_i} - \frac{y_{ij}}{y_{rj}}x_{B_r}\ \forall i = 1,2,\ldots,m, i \neq r$ and $\hat{x}_{B_r} = \frac{x_{B_r}}{y_{rj}}$.

Let $\hat{B} = [\boldsymbol{\beta}_1, \boldsymbol{\beta}_2,\ldots,\boldsymbol{\beta}_{r-1}, \boldsymbol{a}_j, \boldsymbol{\beta}_{r+1},\ldots,\boldsymbol{\beta}_m]$. Then $(\hat{\boldsymbol{x}}_{\hat{B}}, \overbrace{0,0,\ldots,0}^{n-m})^T$ is a BFS of LPP (3) with corresponding basis matrix $\hat{B}$ if

$$\hat{x}_{\hat{B}_i} = x_{B_i} - \frac{y_{ij}}{y_{rj}}x_{B_r} \geq 0\ \forall i = 1,2,\ldots,m. \tag{8}$$

Note that $\hat{x}_{\hat{B}_r} \geq 0$. So, we have to focus only on $i \neq r$. Now if $y_{ij} \leq 0$ for $i \neq r$, then for all such indices i, we have $\hat{x}_{\hat{B}_i} \geq 0$. If $y_{ij} > 0, i \neq r$, then (8) holds if and only if $\frac{x_{B_i}}{y_{ij}} \geq \frac{x_{B_r}}{y_{rj}}$, for all such $i \neq = r$, for which $y_{ij} > 0$. Choosing an index r such that

$$\frac{x_{B_r}}{y_{rj}} = \min\left\{\frac{x_{B_i}}{y_{ij}} : y_{ij} > 0\right\} \tag{9}$$

will make sure that (8) holds.

Now we compute the objective function value at the new BFS $(\hat{\boldsymbol{x}}_{\hat{B}}, \overbrace{0,0,\ldots,0}^{n-m})^T$. Let it be denoted by $\hat{z}$. Then

$$\begin{aligned}
\hat{z} &= \boldsymbol{c}_{\hat{B}}\boldsymbol{x}_{\hat{B}} \\
&= \sum_{i=1}^{m} c_{\hat{B}_i}x_{\hat{B}_i} \\
&= \sum_{i=1,i\neq r}^{m} c_{B_i}\left(x_{B_i} - \frac{y_{ij}}{y_{rj}}x_{B_r}\right) + c_j\frac{x_{B_r}}{y_{rj}} \\
&= \sum_{i=1}^{m} c_{B_i}\left(x_{B_i} - \frac{y_{ij}}{y_{rj}}x_{B_r}\right) + c_j\frac{x_{B_r}}{y_{rj}} \\
&= \sum_{i=1}^{m} c_{B_i}x_{B_i} - \left(\sum_{i=1}^{m} c_{B_i}y_{ij} - c_j\right)\frac{x_{B_r}}{y_{rj}} \\
&= z - (z_j - c_j)\frac{x_{B_r}}{y_{rj}}
\end{aligned}$$

where z is the objective function value at the old BFS $(\boldsymbol{x}_B, \overbrace{0,0,\ldots,0}^{n-m})^T$. If $x_{B_r} > 0$,then $\hat{z} > z$ provided $z_j - c_j < 0$.

Thus, in absence of degeneracy, the objective function value $\hat{z}$ at new BFS $(\hat{\boldsymbol{x}}_{\hat{B}}, \overbrace{0,0,\ldots,0}^{n-m})^T$ strictly improves (since the LPP is a maximization problem) over the current objective function value z at $(\boldsymbol{x}_B, \overbrace{0,0,\ldots,0}^{n-m})^T$ provided we can make sure that $z_j - c_j < 0$.

It may be noted that any value of j satisfying $z_j - c_j < 0$ can be selected for improvement and there is no way to select the unique one out of many, yet it is a numerical convention to choose the one for which $z_j - c_j$ is the most negative in anticipation that it will yield large improvement in the objective function value. The variable x_j for which $z_j - c_j$ is most negative is thus selected to enter in the current basis matrix B and the variable x_{B_r} which is presently a basic variable is selected to leave the basic matrix B by (9). The method in (9) of determining the departing variable index r is called minimum ratio rule. The variable x_j is refereed to as *entering variable* while the variable x_{B_r} leaving the basis is called the *departing variable*.

This iterative scheme of generating improved BFS can be continued till we reach one of the following situations.

(i) $z_j - c_j \geq 0, \forall j = 1, 2, \ldots, n$.

(ii) $z_j - c_j < 0$, for at least one j with $y_{ij} \leq 0$, $\forall i = 1, 2, \ldots, m$.

If case (i) holds then the LPP reaches its optimal solution and the BFS yielded in the last iteration is the optimal solution. Case (ii) indicates the situation when the LPP has an unbounded solution.

4.2 Constraints involving slack variables only

If the constraints of the LPP (3) involves slack variables only, then we can find a BFS instantly, by choosing the slack variables as basic variables and all other variables as non-basic variables, provided all $b_i \geq 0$ $\forall i = 1, 2, \ldots, m$. In this case, the basis matrix B constituted by the columns corresponding to the slack variables becomes the identity matrix and so $\boldsymbol{x}_B = B^{-1}\boldsymbol{b} = \boldsymbol{b}$ and $\boldsymbol{y}_j = B^{-1}\boldsymbol{a}_j = \boldsymbol{a}_j$ $\forall j = 1, 2, \ldots, n$. We summarise the procedure as follows.

1. Start with the inititial BFS obtained by considering slack variables as basic variables. Order of the columns iin the basis matrix will be according to their position in the standard basis.

2. Compute all $z_j - c_j$s. If either of the stopping criteria is not reached, choose the variable corresponding to the most negative $z_j - c_j$ value as entering variable. Determine the departing variable according to minimum ratio rule. The element lying in the junction of the column corresponding to the entering variable and the row corresponding to the departing variable is called *key element*.

3. To transform the column corresponding to the entering variable to a vector of the standard basis, apply elementary row operations to make the key element 1 and all elements below and above it 0.

4. Repeat the above two steps until either of the stopping criteria is reached.

Example 4 *Solve the following LPP by Simplex method:*

$$\begin{aligned} \max z = 15x_1 + 10x_2 & & \\ \textit{subject to}\; 8x_1 + 5x_2 &\leq & 60 \\ 4x_1 + 5x_2 &\leq & 40 \\ x_1, x_2 \geq 0. & & \end{aligned}$$

Solution. Both the constraints are "≤" type with components of the requirement vector positive. Adding the slack variables and associating cost zero with them in the objective function, we have:

$$\begin{aligned} \max z &= 15x_1 + 10x_2 + 0.x_3 + 0.x_4 \\ \text{subject to } 8x_1 + 5x_2 + x_3 + 0.x_4 &= 60 \\ 4x_1 + 5x_2 + 0.x_3 + x_4 &= 40 \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

Observe that $\boldsymbol{b} = \begin{pmatrix} 60 \\ 40 \end{pmatrix} \geq 0$. So, we choose the initial basis as $B = [\boldsymbol{a}_3, \boldsymbol{a}_4] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ with x_3, x_4 being the basic variables. Following are the iterations of Simplex method for the problem in tabular form.

		$\boldsymbol{c} \rightarrow$	15	10	0	0	
$\boldsymbol{c}_B$	$\boldsymbol{x}_B$	$B^{-1}\boldsymbol{b}$	$\boldsymbol{y}_1$	$\boldsymbol{y}_2$	$\boldsymbol{y}_3$	$\boldsymbol{y}_4$	Min. ratio
0	x_3	60	$\boxed{8}$	5	1	0	$15/2 \rightarrow$
0	x_4	40	4	5	0	1	10
$z_j - c_j$		$z = 0$	$-15 \uparrow$	-10	0	0	
15	x_1	$\frac{15}{2}$	1	$\frac{5}{8}$	$\frac{1}{8}$	0	10
0	x_4	10	0	$\boxed{\frac{5}{2}}$	$-\frac{1}{2}$	1	$\frac{20}{3} \rightarrow$
$z_j - c_j$		$z = \frac{225}{2}$	0	$-\frac{5}{8} \uparrow$	$\frac{15}{8}$	0	
15	x_1	5	1	0	$\frac{1}{4}$	$-\frac{1}{4}$	
10	x_2	4	0	1	$-\frac{1}{5}$	$\frac{2}{5}$	
$z_j - c_j$		$z = 115$	0	0	$\frac{7}{4}$	$\frac{1}{4}$	

Since $z_j - c_j \geq 0$ for all $j = 1, 2, 3, 4$, optimality condition is reached and the optimal solution is $x_1 = 5, x_2 = 4, z_{\max} = 115$.

4.3 Constraints involving slack and surplus variables

Unlike the previous example, it is possible that the legitimate variables (original, slack and surplus) in the standard form of LPP fail to posses an identity matrix. In such cases a BFS can not be found instantly. Rather, we search for a basis matrix which gives a feasible solution and begin Simplex method with this initial BFS.

Example 5 *Solve the following LPP by Simplex method:*

$$\begin{aligned} \max z &= x_1 + x_2 \\ \textit{subject to } 2x_1 - x_2 &\leq 10 \\ x_1 + 2x_2 &\geq 4 \\ x_1, x_2 &\geq 0. \end{aligned}$$

Solution. Introducing slack and surplus variables and associating cost zero with them in the objective function, we have:

$$\begin{aligned} \max z &= x_1 + x_2 + 0.x_3 + 0.x_4 \\ \text{subject to } 2x_1 - x_2 + x_3 + 0.x_4 &= 10 \\ x_1 + 2x_2 + 0.x_3 - x_4 &= 4 \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

We see that the coefficient matrix does not contain $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ as a submatrix. So, we can not obtain an initial BFS instantly. Here $\boldsymbol{a}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\boldsymbol{a}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $\boldsymbol{a}_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\boldsymbol{a}_4 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 10 \\ 4 \end{pmatrix}$. We can choose $B = [\boldsymbol{a}_1, \boldsymbol{a}_3] = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ as a basis since $\det(B) = -1 \neq 0$. Now $\begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = B^{-1}\boldsymbol{b} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \cdot \begin{pmatrix} 10 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \geq 0$. So, $B = [\boldsymbol{a}_1, \boldsymbol{a}_3]$ gives a BFS with x_3, x_4 being the basic variables. $\boldsymbol{y} - i$s in the first table are given by $\boldsymbol{y}_1 = B^{-1}\boldsymbol{a}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\boldsymbol{y}_2 = B^{-1}\boldsymbol{a}_2 = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$, $\boldsymbol{y}_3 = B^{-1}\boldsymbol{a}_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\boldsymbol{y}_4 = B^{-1}\boldsymbol{a}_4 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. Following are the further iterations.

		$\boldsymbol{c} \rightarrow$	1	1	0	0	
$\boldsymbol{c}_B$	$\boldsymbol{x}_B$	$B^{-1}\boldsymbol{b}$	$\boldsymbol{y}_1$	$\boldsymbol{y}_2$	$\boldsymbol{y}_3$	$\boldsymbol{y}_4$	Min. ratio
1	x_1	4	1	2	0	-1	-
0	x_3	2	0	-5	1	$\boxed{2}$	$2/2 = 1 \rightarrow$
$z_j - c_j$		$z = 4$	0	1	0	$-1 \uparrow$	
1	x_1	5	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	
0	x_4	1	0	$-\frac{5}{2}$	$\frac{1}{2}$	1	
$z_j - c_j$		$z = 5$	0	$-\frac{3}{2} \uparrow$	$\frac{1}{2}$	0	

In the last table $z_2 - c_2 < 0$ but y_{12} and y_{22} are both negative. Hence the LPP has an unbounded solution.

5 Big-M Method

As in Example (5), when the standard form of an LPP fails to posses an identity matrix, we forcefully or artificially create identity matrix introducing additional variables whenever they are absolutely required for so. These additional variables are assumed to be non-negative and are called *artificial variables.* We consider the cost of these variables in the objective functions as $-M$ where, M is a large positive real number, so that these variables become departing variable as soon as possible in order to maximize the objective function value. If optimality condition is reached, i.e., $z_j - c_j \geq 0 \ \forall j = 1, 2, \ldots, n$ but some artificial variable is still a basic variable, then the LPP has no feasible solution.

Example 6 *Solve the following LPP by Simplex method:*

$$\begin{aligned} \max z = 2x_1 + 5x_2 - 4x_3 & \\ \textit{subject to } x_1 + 2x_2 + x_3 &\leq 4 \\ -2x_1 + 3x_2 &\geq 1 \\ 5x_1 + 4x_2 + 5x_3 &= 7 \\ x_1, x_2, x_3 \geq 0. & \end{aligned}$$

Solution. After introducing slack, surplus and artificial variables, the LPP becomes:

$$\begin{aligned} \max z = 2x_1 + 5x_2 - 4x_3 + 0.x_4 + 0.x_5 - M.x_6 - Mx_7 & \\ \text{subject to } x_1 + 2x_2 + x_3 + x_4 + 0.x_5 + 0.x_6 + 0.x_7 &= 4 \\ -2x_1 + 3x_2 + 0.x_3 + 0.x_4 - x_5 + x_6 + 0.x_7 &= 1 \\ 5x_1 + 4x_2 + 5x_3 + 0.x_4 + 0.x_5 + 0.x_6 + x_7 &= 7 \\ x_1, x_2, \ldots, x_7 \geq 0. & \end{aligned}$$

Observe that $\boldsymbol{b} = \begin{pmatrix} 4 \\ 1 \\ 7 \end{pmatrix} \geq 0$. So, we choose the initial basis as $B = [\boldsymbol{a}_4, \boldsymbol{a}_6, \boldsymbol{a}_7] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ with x_4, x_6 and x_7 being the basic variables. Following are the iteration tables.

		$\boldsymbol{c} \rightarrow$	2	5	-4	0	0	$-M$	$-M$	
$\boldsymbol{c}_B$	$\boldsymbol{x}_B$	$B^{-1}\boldsymbol{b}$	$\boldsymbol{y}_1$	$\boldsymbol{y}_2$	$\boldsymbol{y}_3$	$\boldsymbol{y}_4$	$\boldsymbol{y}_5$	$\boldsymbol{y}_6$	$\boldsymbol{y}_7$	Min. ratio
0	x_4	4	1	2	1	1	0	0	0	2
$-M$	x_6	1	-2	$\boxed{3}$	0	0	-1	1	0	$\frac{1}{3} \rightarrow$
$-M$	x_7	7	5	4	5	0	0	0	1	$\frac{7}{5}$
$z_j - c_j$		$z = 0$	$-3M-2$	$-7M-5 \uparrow$	$-5M+4$	0	M	0	0	
0	x_4	$\frac{10}{3}$	$\frac{7}{3}$	0	-1	1	$\frac{2}{3}$	-	0	$\frac{10}{7}$
5	x_2	$\frac{1}{3}$	$-\frac{2}{3}$	1	0	0	$-\frac{1}{3}$	-	0	-
$-M$	x_7	$\frac{17}{3}$	$\boxed{\frac{23}{3}}$	0	5	0	$\frac{4}{3}$	-	1	$\frac{17}{23} \rightarrow$
$z_j - c_j$		$z = -\frac{17}{3}M + \frac{5}{3}$	$-\frac{23}{3}M - \frac{16}{3} \uparrow$	0	$-5M+4$	0	$-\frac{4}{3}M - \frac{5}{3}$	-	0	
0	x_4	$\frac{37}{23}$	0	0	$-\frac{58}{23}$	1	$\frac{6}{23}$	-	-	$\frac{37}{6}$
5	x_2	$\frac{11}{23}$	0	1	$\frac{10}{23}$	0	$-\frac{5}{23}$	-	-	-
2	x_3	$\frac{17}{23}$	1	0	$\frac{15}{23}$	0	$\boxed{\frac{4}{23}}$	-	-	$\frac{17}{4} \rightarrow$
$z_j - c_j$		$z = \frac{89}{23}$	0	0	$\frac{172}{23}$	0	$-\frac{17}{23} \uparrow$	-	-	
0	x_4	$\frac{1}{2}$	$-\frac{6}{4}$	0	$-\frac{7}{2}$	0	0	-	-	
5	x_2	$\frac{7}{4}$	$\frac{5}{4}$	1	$\frac{5}{4}$	1	0	-	-	
0	x_5	$\frac{17}{4}$	$\frac{4}{23}$	0	$\frac{15}{4}$	0	1	-	-	
$z_j - c_j$		$z = \frac{35}{4}$	$\frac{17}{4}$	0	$\frac{41}{4}$	0	0	-	-	

Since $z_j - c_j \geq 0 \; \forall \, j = 1, 2, \ldots, 7$, optimality criteria is reached and the optimal solution is $x_1 = 0, x_2 = \dfrac{7}{4}, x_3 = 0$ and $z_{\max} = \dfrac{35}{4}$.

It should be noticed that we have dropped the columns corresponding to an artificial variable once it becomes a departing variable, as it is not expected to get an artificial variable as a basic variable in the final table.

6 Two Phase Method

There is another way to tackle the artificial variables in an LPP. As evident, our initial goal is to remove the artificial variables which were introduced only to get the initial BFS of the given LPP, and thereafter move ahead to optimize the original objective function. It is the optimal solution of LPP (provided if exists) that is primarily to be obtained. To attain this goal, we divide the process in two phase. In Phase – I, we focus mainly on the artificial variables and their removal from the system, while in Phase – II we concentrate on the original objective function and optimize it. This led to another method of solving LPP called the Two Phase Method.

In Phase – I, we construct a new objective function by associating zero cost to the slack and surplus variables and cost -1 with all artificial variables. Two cases may arise in the last table of Phase – I.

(i) If one or more artificial variable is present in the final table at positive level (i.e., having positive value), but the optimality criteria $z_j - c_j \geq 0$ for all j holds, then the given LPP has no feasible solution.

(ii) If in the last table of Phase – I, either no artificial variable is present, or some artificial variables are present at zero value, then we have obtained at least one feasible solution of the original LPP. Now, if no artificial variable appears in the optimal table of Phase – I (i.e., all artificial variables are non-basic and hence taking values zero), we actually have an initial BFS of the given LPP in our hand and we move to the next phase (i.e., Phase –II) with initial BFS of the original LPP. However, if one or more artificial variable is present in the optimal basis of the Phase – I with zero, then we have only a feasible solution of the given LPP in our hand and not the BFS because the number of legitimate basic variables is then strictly less than m (not counting the artificial ones in the optimal basis). Nevertheless, we move to Phase – II in this case too.

In Phase – II, we consider the objective function of the given LPP and make all necessary changes in the last optimal table of the Phase – I to take care of the cost coefficients in the original objective function. Then we move ahead with the simplex iterations to achieve the optimal solution (if it exists) of the given LPP. Now while solving the LPP in Phase – II, we have to be cautious if any artificial variable is present in the basis at zero level. If at any subsequent iteration of the simplex method, any of the artificial variable present in the current basis starts becoming positive then that artificial variable should be immediately removed from the basis by choosing it the departing variable.

Example 7 *Solve the following LPP by Two Phase method:*

$$\begin{aligned} \max z = 2x_1 - x_2 + x_3 & & \\ \textit{subject to } x_1 + x_2 - 3x_3 &\leq& 8 \\ 4x_1 + x_2 - 3x_3 &\geq& 2 \\ 2x_1 + 3x_2 - x_3 &\geq& 4 \\ x_1, x_2, x_3 \geq 0. & & \end{aligned}$$

Solution. After introducing slack, surplus and artificial variables, the constraints become:

$$\begin{aligned} x_1 + x_2 - 3x_3 + x_4 + 0.x_5 + 0.x_6 + 0.x_7 + 0x_8 &=& 8 \\ 4x_1 + 2x_2 - 3x_3 + 0.x_4 - x_5 + 0.x_6 + x_7 + 0.x_8 &=& 2 \\ 2x_1 + 3x_2 - x_3 + 0.x_4 + 0.x_5 - x_6 + 0.x_7 + x_8 &=& 4 \\ x_1, x_2, \ldots, x_8 \geq 0. & & \end{aligned}$$

Thus in Phase – I, we consider the LPP

$$\begin{aligned} \max z = -x_7 - x_8 & & \\ \text{subject to } x_1 + x_2 - 3x_3 + x_4 + 0.x_5 + 0.x_6 + 0.x_7 + 0x_8 &=& 8 \\ 4x_1 + 2x_2 - 3x_3 + 0.x_4 - x_5 + 0.x_6 + x_7 + 0.x_8 &=& 2 \\ 2x_1 + 3x_2 - x_3 + 0.x_4 + 0.x_5 - x_6 + 0.x_7 + x_8 &=& 4 \\ x_1, x_2, \ldots, x_8 \geq 0. & & \end{aligned}$$

Observe that $\boldsymbol{b} = \begin{pmatrix} 8 \\ 2 \\ 4 \end{pmatrix} \geq 0$. So, we choose the initial basis as $B = [\boldsymbol{a}_4, \boldsymbol{a}_7, \boldsymbol{a}_8] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ with x_4, x_7 and x_8 being the basic variables. Following are the iteration tables.

		$c \rightarrow$	0	0	0	0	0	0	-1	-1	
c_B	x_B	$B^{-1}b$	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	Min. ratio
0	x_4	8	1	1	-3	1	0	0	0	0	8
-1	x_7	2	$\boxed{4}$	-1	1	0	-1	0	1	0	$\frac{1}{2} \rightarrow$
-1	x_8	4	2	3	-1	0	0	-1	0	1	2
$z_j - c_j$			$-6\uparrow$	-2	0	0	1	1	0	0	
0	x_4	$\frac{15}{2}$	0	$\frac{5}{4}$	$-\frac{13}{4}$	1	$\frac{1}{4}$	0	-	0	6
0	x_1	$\frac{1}{2}$	1	$-\frac{1}{4}$	$\frac{1}{4}$	0	$-\frac{1}{4}$	0	-	0	-
-1	x_8	3	0	$\boxed{\frac{7}{2}}$	$-\frac{3}{2}$	0	$-\frac{1}{2}$	-1	-	1	$\frac{6}{5} \rightarrow$
$z_j - c_j$			0	$-\frac{5}{2}\uparrow$	$\frac{3}{2}$	0	$-\frac{1}{2}$	1	-	0	
0	x_4	$\frac{45}{7}$	0	0	$-\frac{18}{7}$	1	$\frac{1}{14}$	$\frac{5}{14}$	-	-	
0	x_1	$\frac{5}{7}$	1	0	$\frac{1}{7}$	0	$-\frac{3}{14}$	$-\frac{1}{14}$	-	-	
0	x_2	$\frac{6}{7}$	0	1	$-\frac{3}{7}$	0	$\frac{1}{7}$	$-\frac{2}{7}$	-	-	
$z_j - c_j$			0	0	0	0	0	0	-	-	

We observe that no artificial variable is present in the basis in the last table of Phase – I above. So we move to Phase – II. The final table of Phase – I is taken as the initial table of Phase – II with cost coefficients being replaced by original cost coefficients. Iterations in Phase - II are as follows.

		$c \rightarrow$	2	-1	1	0	0	0	
c_B	x_B	$B^{-1}b$	y_1	y_2	y_3	y_4	y_5	y_6	Min. ratio
0	x_4	$\frac{45}{7}$	0	0	$-\frac{18}{7}$	1	$\frac{1}{14}$	$\frac{5}{14}$	90
2	x_1	$\frac{5}{7}$	1	0	$\frac{1}{7}$	0	$-\frac{3}{14}$	$-\frac{1}{14}$	-
-1	x_2	$\frac{6}{7}$	0	1	$-\frac{3}{7}$	0	$\boxed{\frac{1}{7}}$	$-\frac{2}{7}$	$6 \rightarrow$
$z_j - c_j$			0	0	$-\frac{2}{7}$	0	$-\frac{4}{7}\uparrow$	$\frac{1}{7}$	
0	x_4	6	0	$-\frac{1}{2}$	$-\frac{5}{2}$	1	0	$\frac{1}{2}$	
2	x_1	2	1	$\frac{3}{2}$	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	
0	x_5	6	0	7	-3	0	1	-2	
$z_j - c_j$			0	3	-2	0	0	-1	

In the last table of Phase – II, $z_3 - c_3 < 0$, but $y_{i3} < 0\ \forall i = 1,2,3$. Thus the given LPP has an unbounded solution.