Hilbert Spaces

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1 Inner Product Space

Definition 1.1. Let X be a linear space over a field K. An inner product on X is a function $\langle\ ,\ \rangle$ from $X \times X$ to K such that for all x, y, z in X and k in K, we have

(i) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$, (positive-definiteness),

(ii) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ and $\langle kx, y\rangle = k\langle x, y\rangle$, (linearity in the first variable),

(iii) $\langle y, x\rangle = \overline{\langle x, y\rangle}$,(conjugate-symmetry).

An inner product space is a linear space with an inner product defined on it.
From the axioms above one can immediately derive

1. $\langle k_1x + k_2y, z\rangle = k_1\langle x, z\rangle + k_2\langle y, z\rangle$,
2. $\langle x, k_1y + k_2z\rangle = \overline{k_1}\langle x, y\rangle + \overline{k_2}\langle x, z\rangle$, i.e., an inner-product is conjugate-linear in the second variable,

for all x, y, z in X and k_1, k_2 in K.

An inner product on X defines a norm on X given by

$$\|x\| = \sqrt{\langle x, x\rangle}$$

and a metric on X given by

$$d(x, y) = \|x - y\| = \sqrt{\langle x - y, x - y\rangle}.$$

Example 1.2. Some examples of inner product spaces are

(i) The space $\mathbb{R}^n$ with inner product defined by

$$\langle x, y\rangle = x(1)y(1) + \cdots + x(n)y(n),$$

where $x = (x(1), \ldots, x(n))$ and $y = (y(1), \ldots, y(n))$.
The norm is given by

$$\|x\| = \langle x, x\rangle^{1/2} = (x(1)^2 + \cdots + x(n)^2)^{1/2}.$$

(ii) The space $\mathbb{C}^n$ with inner product defined by

$$\langle x, y\rangle = x(1)\overline{y(1)} + \cdots + x(n)\overline{y(n)},$$

where $x = (x(1), \dots, x(n))$ and $y = (y(1), \dots, y(n))$.
Here $\|x\| = (x(1)\overline{x(1)} + \cdots + x(n)\overline{x(n)})^{1/2} = (|x(1)|^2 + \cdots + |x(n)|^2)^{1/2}$.

(iii) The space l^2 with inner product defined by

$$\langle x, y\rangle = \sum_{j=1}^{\infty} x(j)\overline{y(j)},$$

for $x = (x(1), x(2), \dots)$ and $y = (y(1), y(2), \dots)$.
The norm is defined by
$\|x\| = \langle x, x\rangle^{1/2} = (\sum_{j=1}^{\infty} |x(j)|^2)^{1/2}$.

(iv) $C[a, b]$ is an inner product space with inner product defined by

$$\langle x, y\rangle = \int_a^b x(t)\overline{y(t)}dt.$$

(v) $C^1[a, b]$, the linear space of all scalar-valued continuously differentiable functions on $[a, b]$, is an inner product space with inner product defined by

$$\langle x, y\rangle = x(a)\bar{y}(a) + \int_a^b x'(t)\bar{y}'(t)dt.$$

The norm obtained from an inner product on a linear space X satisfies the parallelogram equality:

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2),$$

for any two members x, y in X. (Verify!)

So every inner product space is a normed space but the opposite is not true. If a norm does not satisfy parallelogram equality, it cannot be obtained from an inner product.

Example 1.3. We now give examples of normed spaces which are not inner product spaces.

(i) The space l^p with $p \neq 2$.
Let us take $x = (1, 1, 0, 0, \cdots), y = (1, -1, 0, 0, \cdots)$ in $l^p, p \neq 2$. Then $\|x\| = \|y\| = 2^{1/p}$ and $\|x + y\| = \|x - y\| = 2$ so that parallelogram equality is not satisfied.

(ii) The space $C[a, b]$ with norm $\|x\| = \max_{t\in[a,b]} |x(t)|$.
If we take $x(t) = 1$ and $y(t) = (t - a)/(b - a)$, we have $\|x\| = \|y\| = 1$ whereas $\|x + y\| = 2, \|x - y\| = 1$. The parallelogram equality is not satisfied.

Lemma 1.4. *(Polarization identity) Let X be an inner product space. We have*

$$4\langle x, y\rangle = \langle x+y, x+y\rangle - \langle x-y, x-y\rangle + i\langle x+iy, x+iy\rangle - i\langle x-iy, x-iy\rangle$$

for all $x, y \in X$.

Let us now see whether $\|\cdot\|$ defined from the inner product $\langle\ ,\ \rangle$ indeed satisfies all the axioms of a norm. In fact, if x, y belong to the inner product space X and $k \in K$ then

(i) $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$ (from (i) of definition of inner product).

(ii) $\|kx\|^2 = \langle kx, kx\rangle = k\bar{k}\langle x, x\rangle = |k|^2\|x\|^2$ yields $\|kx\| = |k|\|x\|$.

(iii) In order to prove triangle inequality

$$\|x+y\| \leq \|x\| + \|y\|$$

we establish

Lemma 1.5. *(Schwarz inequality) For all $x, y \in X$,*

$$|\langle x, y\rangle| \leq \|x\|\|y\|,$$

where equality holds if and only if $\{x, y\}$ is a linearly dependent set.

Proof. Without loss of generality, we take $y \neq 0$ (taking $y = 0$, LHS = RHS (check!)). Let k be any scalar. Then

$$0 \leq \|x - ky\|^2 = \langle x - ky, x - ky\rangle = \langle x, x\rangle - \bar{k}\langle x, y\rangle - k[\langle y, x\rangle - \bar{k}\langle y, y\rangle].$$

The expression in the brackets $[\cdots]$ is zero if we choose $\bar{k} = \langle y, x\rangle / \langle y, y\rangle$. Then the inequality reduces to

$$0 \leq \langle x, x\rangle - \frac{\langle y, x\rangle}{\langle y, y\rangle}\langle x, y\rangle = \|x\|^2 - \frac{|\langle x, y\rangle|^2}{\|y\|^2},$$

whence we obtain the required inequality.

Equality holds if and only if $y = 0$ or $\|x - ky\|^2 = 0$ so that $x = ky$ and the set $\{x, y\}$ becomes linearly dependent. □

Now we prove triangle inequality using Schwarz inequality. We have

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + \langle x, y\rangle + \langle y, x\rangle + \|y\|^2.$$

By triangle inequality for numbers, we get

$$\begin{aligned} \|x+y\|^2 &\leq \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2. \end{aligned}$$

Hence $\|x+y\| \leq \|x\| + \|y\|$.

Lemma 1.6. *(Continuity of inner product) Let X be an inner product space. If $x_n \to x$ and $y_n \to y$ in X then $\langle x_n, y_n\rangle \to \langle x, y\rangle$ in K.*

Proof. We have

$$\begin{aligned} |\langle x_n, y_n\rangle - \langle x, y\rangle| &= |\langle x_n, y_n\rangle - \langle x_n, y\rangle + \langle x_n, y\rangle - \langle x, y\rangle| \\ &\leq |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle| \\ &\leq \|x_n\|\|y_n - y\| + \|x_n - x\|\|y\| \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

□

If a norm $\|\cdot\|$ on a linear space X satisfies parallelogram equality, then an inner product $\langle\ ,\ \rangle$ can be defined on X by polarization identity,

$$\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2)$$

which satisfies $\langle x, x\rangle^{1/2} = \|x\|$. (Prove!)

A Hilbert space H is a complete inner product space (complete in the metric defined by the inner product). The spaces $\mathbb{R}^n, \mathbb{C}^n$ and l^2 are Hilbert spaces with inner products defined as above. We find in Example 1.3 examples of Banach spaces which are not Hilbert spaces.

2 Orthonormal Sets in a Hilbert Space

Let X be an inner product space. At this stage, we know the concepts of orthogonal and orthonormal sets in X and

1. Pythagorean relation for an orthogonal set in X,
2. An orthonormal set is linearly independent,
3. If E is an orthonormal subset of X, the $\|x - y\| = \sqrt{2}$ for all $x \neq y$ in E.
4. Gram-Schimdt orthonormalization process for constructing an orthonormal sequence from a linearly independent sequence (x_j), and
5. Bessel inequality : If $\{u_1, u_2, \cdots\}$ is a countable orthonormal set in an inner product space X, then for every $x \in X$,
$$\sum_{n=1}^{\infty} |\langle x, u_n\rangle|^2 \leq \|x\|^2.$$

Now let H be a Hilbert space and (u_k) be an orthonormal sequence in H. We consider a series $\sum_{k=1}^{\infty} c_k u_k$ in H. Let (s_n) be the sequence of partial sums of the series, $s_n = c_1 u_1 + \cdots + c_n u_n$. Then we have

Theorem 2.1. *The series $\sum_{k=1}^{\infty} c_k u_k$ converges if and only if the series $\sum_{k=1}^{\infty} |c_k|^2$ converges.*

Proof. Let (s'_n) be the sequence of partial sums of the series $\sum_{k=1}^{\infty} |c_k|^2$. Then for $n > m$,

$$\begin{aligned}
\|s_n - s_m\|^2 &= \|c_{m+1}u_{m+1} + \cdots + c_n u_n\|^2 \\
&= \langle c_{m+1}u_{m+1} + \cdots + c_n u_n, c_{m+1}u_{m+1} + \cdots + c_n u_n\rangle \\
&= |c_{m+1}|^2 + \cdots + |c_n|^2 \\
&= s'_n - s'_m.
\end{aligned}$$

So the sequence (s_n) is cauchy in H (which is complete) if and only if the sequence (s'_n) is cauchy in $\mathbb{R}$(which is again complete). Hence the result. □

Theorem 2.2. *Suppose the series $\sum_{k=1}^{\infty} c_k u_k$ is convergent and has the sum x. Then the coefficients c_k are the Fourier coefficients $\langle x, u_k\rangle$ and so we write $x = \sum_{k=1}^{\infty}\langle x, u_k\rangle u_k$.*

From Bessel inequality and Theorem 1.1, we obtain

Theorem 2.3. *The series $\sum_{k=1}^{\infty}\langle x, u_k\rangle u_k$ converges for any x in H.*

Definition 2.4. In a Hilbert space H an orthonormal system (u_k) is called an orthonormal basis if there is no orthonormal system in H to contain (u_k) as a proper subset.

Theorem 2.5. *Let (u_k) be an orthonormal sequence in H. Then the following conditions are equivalent:*

(i) (u_k) is an orthonormal basis.

(ii) $\langle x, u_k\rangle = 0, \forall k$ implies $x = 0$.

(iii) $x = \sum_{k=1}^{\infty}\langle x, u_k\rangle u_k$ for each $x \in H$. (Fourier expansion)

(iv) $\sum_{k=1}^{\infty} |\langle x, u_k\rangle|^2 = \|x\|^2$ for every $x \in H$. (Parseval formula)

Proof. $(i) \Rightarrow (ii)$. Suppose we find a non-zero x in H such that $\langle x, u_k\rangle = 0, \forall k$. Let $u = \frac{x}{\|x\|}$, then $\|u\| = 1$ and $\langle u, u_k\rangle = 0, \forall k$. So we get an orthonormal system $(u_k) \cup (u)$ properly containing (u_k), contradicting the maximality of the orthonormal set (u_k). Thus (ii) holds.

$(ii) \Rightarrow (iii)$. Let s_n be the sequence of partial sums of the series $\sum_{k=1}^{\infty}\langle x, u_k\rangle u_k$. Then $\langle s_n, u_k\rangle = \langle x, u_k\rangle$ for $1 \le k \le n$. Now for all j

$$\langle x - \sum_{k=1}^{\infty}\langle x, u_k\rangle u_k, u_j\rangle = \langle x, u_j\rangle - \lim_{n\to\infty}\langle s_n, u_j\rangle = \langle x, u_j\rangle - \langle x, u_j\rangle = 0,$$

by continuity of inner product. By (ii), $x - \sum_{k=1}^{\infty}\langle x, u_k\rangle u_k = 0$.

$(iii) \Rightarrow (iv)$. We have

$$\begin{aligned}
\|x\|^2 = \langle x, x\rangle &= \lim_{n\to\infty}\langle \sum_{k=1}^{n}\langle x, u_k\rangle u_k, \sum_{j=1}^{n}\langle x, u_j\rangle u_j\rangle \\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\langle x, u_k\rangle\overline{\langle x, u_k\rangle} \\
&= \lim_{n\to\infty}\sum_{k=1}^{n} |\langle x, u_k\rangle|^2 \\
&= \sum_{k=1}^{\infty} |\langle x, u_k\rangle|^2.
\end{aligned}$$

$(iv) \Rightarrow (i)$. Suppose there is an orthonormal system strictly containing (u_k), say, $\{u, u_1, u_2, \cdots\}$. By (iv),

$$\|u\|^2 = \sum_{k=1}^{\infty} |\langle u, u_k \rangle|^2 = 0,$$

a contradiction, since $\|u\| = 1$. So (i) is proved. □

Theorem 2.6. *In a separable Hilbert space H, every orthonormal set is countable.*

Proof. Let S be any countable dense set in H and M be an orthonormal set. Let $x, y \in M, x \neq y$. Then $\|x - y\| = \sqrt{2}$. Consider the disjoint spherical neighbourhoods N_x of x and N_y of y, each of radius $\sqrt{2}/3$. As S is dense in H, there is a point $a \in S \cap N_x$ and a point $b \in S \cap N_y$. So there is an one - one correspondence between a pair of distinct elements in M and a pair of distinct elements in H. Thus M is countable. □

3 Orthogonal complements and Projection theorem

A subset A of a vector space X is said to be a convex set if for any $x, y \in A$ and $0 \leq c \leq 1$, it follows that $cx + (1 - c)y \in A$.

Any subspace on an inner product space is always convex. Intersection of convex sets is again convex.

Theorem 3.1. *Let X be an inner product space and $M \neq \phi$ a convex subset which is complete(in the metric induced by the inner product). Then for every given $x \in X$ there exists a unique $y_0 \in M$ such that*

$$\inf_{y \in M} \|x - y\| = \|x - y_0\|.$$

Proof. See [2]. □

If a vector space X is a direct sum of its two subspaces Y and Z, i.e., $X = Y \oplus Z$ then Z is called an algebraic complement of Y in X.
If Y is a subset of an inner product space X, we define $Y^{\perp} = \{x \in X : x \perp Y\}$.

Theorem 3.2. (Projection Theorem)
Let Y be any closed subspace of a Hilbert space H. Then $H = Y \oplus Y^{\perp}$.

Proof. See [2]. □

In the above Projection Theorem, each $x \in H$ has a unique representation $x = y + z$ where $y \in Y$ and $z \perp Y$. Here H is represented as a direct sum a closed subspace Y and its orthogonal complement $Y^{\perp}$. $Y^{\perp}$ is closed because of continuity of inner product function (verify!). y is called the orthogonal projection or projection of x on Y.

The mapping $P : H \to Y$ defined by $Px = y$ is called the orthogonal projection operator or orthogonal projection of H onto Y. Verify that
(i) P is linear,
(ii) P is bounded and $\|P\| \leq 1$. In fact, $\|P\| = 1$,
(iii) P is idempotent, i.e., $P^2 = P$.

(iv) $\langle Px_1, x_2 \rangle = \langle x_1, Px_2 \rangle$, for $x_1, x_2 \in H$.
Note that P maps H onto Y, Y onto itself, $Y^\perp$ onto $\{0\}$. Find null space and range space of P.

Problem. Prove that a subspace Y of a Hilbert space H is closed in H if and only if $Y = Y^{\perp\perp}$.

4 Riesz Theorem on representation of bounded linear functional over Hilbert space H

We would always like to know the general form of a bounded linear functional acting on a given space. Such representation in respect of some normed linear spaces are known, but their derivations are quite complicated. However, situation becomes simple in the setting of Hilbert space H.

Theorem 4.1. (Riesz's theorem)
If f is a bounded linear functional over a Hilbert space H, then $f(x) = \langle x, z \rangle$ for all $x \in H$ and for some $z \in H$ uniquely determined by f such that $\|z\| = \|f\|$.

Proof. See [2] or any other relevant book. □

Problem. What is a sesquilinear form? Do we have a Riesz theorem for general representation of sesquilinear forms on Hilbert spaces? Find out!

5 Hilbert-Adjoint Operator, Self-Adjoint, Unitary and Normal Operators

Study from [2] the following articles:
3.9 Hilbert-Adjoint Operator,
3.10 Self-Adjoint, Unitary and Normal Operators.

Problem Set (Section 5)

1. Define Hilbert-Adjoint operator. Show that it exists uniquely and is a bounded linear operator. What is the norm of Hilbert-Adjoint operator?

2. For $Q : X \to X$, where X is a complex inner product space, if $\langle Qx, x \rangle = 0, \quad \forall x \in X$, then prove that $Q = 0$.

3. Write the properties of Hilbert-Adjoint operator. Also prove them.

4. If T is a self-adjoint operator in a Hilbert space H, show that for evry natural number n, T^n is self-adjoint.

5. Prove that the class of all self-adjoint operators on H forms a closed real subspace of $BL(H)$, and hence it is a Banach space.

6. Let $T : H \to H$ be a self-adjoint operator. Then prove that

   (i) all eigen values of T (if they exist) are real, and

   (ii) eigen vectors corresponding to different eigen values of T are orthogonal.

References

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[2] E. Kreyszig, Introductory Functional Analysis with Applications, Wiley Student Edition(2017).

[3] P. K. Jain, O. P. Ahuja, Functional Analysis, New Age International Publishers(2014).