

Ideal Fermi Gas

April 11, 2020

1 Introduction

The properties of an “degenerate” ideal Fermi gas are strongly determined by the Pauli principle in the limit:

$$kT \gg \mu; \quad \beta\mu \gg 1 \quad (1)$$

where notations bear usual meaning k being the Boltzman constant and μ is the chemical potential of the system. We make following assumptions:

- Ideal means constituents has no mutual interaction and the constituents are of negligible size.
- Such a system is strictly controlled by the principles of quantum mechanics.

1.1 Grand Canonical Partition Function

Consider a gas of N non-interacting fermions, e.g., electrons, whose single-particle wavefunctions $\phi(r)$ are plane-waves. In this case, a complete set of quantum numbers \vec{j} is given, for instance, by the three cartesian components of the wave vector \vec{k} and the z spin projection m_s of an electron (NOTE that although we are considering a gas of electrons, they can not have coulomb repulsion because of the condition 1 stated above):

$$j \equiv (k_x, k_y, k_z, m_s) \quad (2)$$

The partition function can be written as usual formula $\mathcal{Z} = \sum_j g_j e^{-\beta(\epsilon_j - \mu)}$, where g_j is the degeneracy factor of the j th energy state. The summation over the states j (whenever it has to be performed) can then be reduced to the summation over states with different wavevector \vec{k} with $\vec{p} = \hbar\vec{k}$ as : $\sum_j \Rightarrow (2s + 1) \sum_k$ where the summation over the spin quantum number gives $2s+1$ factor from the spin states $m_s = -s, -s+1, \dots, s$. Now as dictated by the Fermi-Dirac statistics, g_j is either 0 or 1 as the number of electrons in a single energy

state can be either 0 or 1. Hence, $\mathcal{Z} = \sum_j (1 + e^{-\beta(\epsilon_j - \mu)})$. However, it is customary to write partition function in terms of Logarithm

$$\ln \mathcal{Z} = \sum_j \ln(1 + e^{-\beta(\epsilon_j - \mu)}) \quad (3)$$

where the sum is over all single particle energy states.

1.2 Density of states

We assume that the electrons are in a volume defined by a cube with sides L_x, L_y, L_z and volume $V = L_x L_y L_z$. For the one-particle wavefunction $\langle \vec{r} | \vec{k} \rangle = \psi_{\vec{k}}(r) = \frac{e^{i\vec{k} \cdot \vec{r}}}{\sqrt{V}}$. we use periodic boundary condition i.e. along x direction we have $e^{ik_x \cdot x} = e^{ik_x \cdot x + ik_x L_x}$ which is then translated into a condition for the allowed k-values:

$$e^{ik_x L_x} = e^{i2\pi n_x} \quad (4)$$

and hence, we get quantization condition $k_x = \frac{2\pi n_x}{L_x}$. Same happens for y and z directions.

Each state has in k-space an average volume of

$$\Delta k = \frac{(2\pi)^3}{L_x L_y L_z} = \frac{(2\pi)^3}{V} \quad (5)$$

For large $V \rightarrow \infty$, we can replace the sum by integral over all quantum numbers

$$\sum_r \rightarrow (2s+1) \frac{1}{\Delta k} \int d^3 k = (2s+1) \frac{V}{(2\pi)^3} \int d^3 k = (2s+1) \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int \sqrt{\epsilon} d\epsilon \quad (6)$$

Here to go from k-variable to energy variable, we have used the non-relativistic dispersion relation i.e.

$$\epsilon_j = \frac{\hbar^2 k^2}{2m} \quad (7)$$

Homework: Do these variable transformations and check the expressions given above.

We see that the above expression has a simple physical interpretation of density of single-particle states. If we now get back on the expression for the grand canonical partition function, we see that we have to add all possible energy states with their degeneracy factor. This is simply given by

$$\ln \mathcal{Z} = (2s+1) \frac{V}{(2\pi)^3} \int d^3 k \ln \left(1 + \exp \left[-\beta \left(\frac{\hbar^2 k^2}{2m} - \mu \right) \right] \right) \quad (8)$$

$$= (2s+1) \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int \sqrt{\epsilon} d\epsilon \ln (1 + \exp [-\beta(\epsilon - \mu)]) \quad (9)$$

When needed we will use either of these above equations.

2 Grand Canonical Potential

We consider now the expression for the fermionic grand canonical potential $\Omega(T, V, \mu)$ as a function of the volume, the temperature, and the chemical potential:

$$\Omega(T, V, \mu) = -kT \ln \mathcal{Z} \quad (10)$$

Note this potential is just like free energy (in fact the formula resembles with that of the Helmholtz free energy). We will use here the k-integral of eqn (6) written above. Now writing k-integral in spherical-polar coordinates $\int d^3k = 4\pi \int k^2 dk$ we express the above equation as:

$$-\beta\Omega(T, V, \mu) = (2s+1) \frac{V}{(2\pi)^3} 4\pi \int k^2 dk \ln \left(1 + z \exp\left(-\beta \frac{\hbar^2 k^2}{2m}\right) \right) \quad (11)$$

where we have used the usual notations $z = e^{\beta\mu}$ and $\epsilon_k = \frac{\hbar^2 k^2}{2m}$. z is called the fugacity.

In order to perform the integration, we make some variable transformations by introducing some dimensionless variable x

$$x = \hbar k \sqrt{\frac{\beta}{2m}}; \quad k^2 dk = \left(\frac{2m}{\beta\hbar^2}\right)^{3/2} x^2 dx \quad (12)$$

Thus one can write above eqn in terms of x

$$-\beta\Omega(T, V, \mu) = (2s+1) \frac{4V}{\sqrt{\pi}} \left(\frac{m}{2\pi\beta\hbar^2}\right)^{3/2} \int_0^\infty x^2 dx \ln \left(1 + z e^{-x^2} \right) \quad (13)$$

By making use of the definition of the thermal de Broglie wave length $\lambda = \sqrt{\frac{2\pi\beta\hbar^2}{m}}$, we can write it as

$$-\beta\Omega(T, V, \mu) = \frac{(2s+1) 4V}{\lambda^3} \frac{1}{\sqrt{\pi}} \int_0^\infty x^2 dx \ln \left(1 + z e^{-x^2} \right) \quad (14)$$

Now in order to perform the integral term-by term we expand the log term as $\ln(1+y) = \sum_{n=1}^\infty (-1)^{n+1} \frac{y^n}{n}$. Hence, using this formula we get

$$\begin{aligned} \int_0^\infty x^2 dx \ln \left(1 + z e^{-x^2} \right) &= \sum_{n=1}^\infty (-1)^{n+1} \frac{z^n}{n} \int_0^\infty dx x^2 e^{-nx^2} \\ &= \sum_{n=1}^\infty (-1)^{n+1} \frac{z^n}{n} \left(-\frac{d}{dn} \int_0^\infty dx e^{-nx^2} \right) \\ &= \sum_{n=1}^\infty (-1)^{n+1} \frac{z^n}{n} \left(-\frac{d}{dn} \frac{1}{2} \sqrt{\pi} \frac{1}{\sqrt{n}} \right) \end{aligned}$$

where we have used the trick of integrating the γ function by taking derivative of a parameter

$$I(\alpha) = \int_0^\infty e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$$

then, we can use the parameter α to perform integration of similar function multiplied by powers of x

$$\int_0^{\infty} x^2 e^{-\alpha x^2} dx = -\frac{d}{d\alpha} \int_0^{\infty} e^{-\alpha x^2} dx = -\frac{dI(\alpha)}{d\alpha}$$

Therefore, our final result is

$$\int_0^{\infty} x^2 \ln(1 + z e^{-x^2}) dx = \frac{\sqrt{\pi}}{4} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n^{5/2}} \quad (15)$$

We make some rearrangement in the constant terms of the final expression and define

$$f_{5/2}(z) = (-1)^{n+1} \frac{z^n}{n^{5/2}} = \frac{\sqrt{4}}{\pi} \int_0^{\infty} x^2 \ln(1 + z e^{-x^2}) dx \quad (16)$$

this is nothing but to write a complicated looking equation in a compact form. Hence, our final form for the grand canonical potential for an Ideal Fermi gas is,

$$\beta\Omega(T, V, \mu) = -\frac{2s+1}{\lambda^3} V f_{5/2}(z) \quad (17)$$

This generic expression can be used to find all other thermodynamic variables. For example, pressure (P) is related to the grand canonical potential as, $\Omega = -PV$. Hence, we can derive pressure of an Ideal Fermi Gas

$$\frac{P}{kT} = \frac{2s+1}{\lambda^3} V f_{5/2}(z) \quad (18)$$

2.1 Internal Energy

The Fermi-Dirac distribution is the average value of number of particles and is given by

$$\langle n_j \rangle = \frac{1}{\exp[-\beta(\epsilon_j - \mu) + 1]} \quad (19)$$

We can also obtain expressions for the expectation value of energy (which corresponds, in the thermodynamic limit, to the internal energy of the system).

$$U = \sum_j \epsilon_j \langle n_j \rangle = \sum_j \frac{\epsilon_j}{\exp[\beta(\epsilon_j - \mu) + 1]} \quad (20)$$

and the total number of particles:

$$N = \sum_j \langle n_j \rangle = \sum_j \frac{1}{\exp[\beta(\epsilon_j - \mu) + 1]} \quad (21)$$

Here we will use the energy integral as we will see energy integral will be much more easier to solve than the k integral. Simple reason is k is a vector, so it is a volume integral. On the other hand, energy is a scalar, so we have to do only

one integration. Thus we can use the energy integral factor from equation (6) and write

$$U = (2s + 1) \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int \frac{\epsilon^{3/2}}{\exp(\beta(\epsilon - \mu) + 1)} d\epsilon \quad (22)$$

and total number of particles,

$$N = (2s + 1) \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int_0^\infty \frac{\sqrt{\epsilon}}{\exp(\beta(\epsilon - \mu) + 1)} d\epsilon \quad (23)$$

We will use a simple trick to do this integral without actually doing any. Let us write eqn (9) again,

$$\ln \mathcal{Z} = (2s + 1) \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int \sqrt{\epsilon} d\epsilon \ln(1 + \exp[-\beta(\epsilon - \mu)])$$

Performing an integration by parts, we get,

$$\ln \mathcal{Z} = (2s + 1) \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \left(\frac{2\beta}{3} \right) \int \frac{(\epsilon)^{3/2}}{\exp(\beta(\epsilon - \mu) + 1)} d\epsilon \quad (24)$$

Homework: Perform the integration by-parts carefully. What happened to the 1st term of integration ?

Now, note that we almost get the exact expression of internal energy as in eqn (22):

$$\ln \mathcal{Z} = \frac{2}{3} \frac{U}{kT} \quad (25)$$

As we know grand potential $\Omega = -kT \ln \mathcal{Z} = -PV$,

$$\frac{2}{3} \frac{U}{kT} = PV \Rightarrow U = \frac{3}{2} PV \quad (26)$$

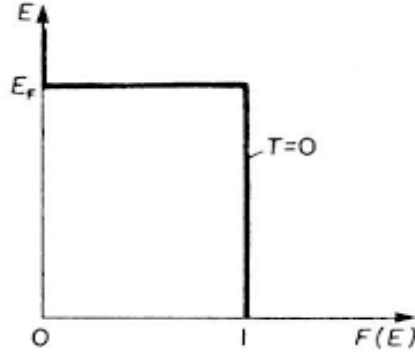
Therefore, we see that the total energy of an ideal (non-relativistic) Fermi gas is exactly equal to the classical monatomic ideal gas of particles !!

3 Completely degenerate Fermi Gas $T \rightarrow 0$

We usually refer to a quantum gas in the ground state, at zero temperature, as completely degenerate. For fermions at zero temperature (or, equivalently $\beta \rightarrow \infty$) the most probable occupation number of the single-particle states is a step function of energy (see figure above). The chemical potential at zero temperature, ϵ_F , is called Fermi energy. In the above figure, we show that all the energy levels are occupied below the Fermi energy (for $\epsilon < \epsilon_F$), and empty for $\epsilon > \epsilon_F$.

Let us calculate the Fermi energy in terms of the number of particles N and volume V . In the limit $\beta \rightarrow \infty$, eqn(23) may be written as

$$N = (2s + 1) \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int_0^{\epsilon_F} \frac{\sqrt{\epsilon}}{\exp(\beta(\epsilon - \mu) + 1)} d\epsilon$$



Note that the upper limit of the integration is up to ϵ_F , since above that energy the occupation number vanishes i.e. states are empty. Since, at (or close to) zero temperature $\mu = \epsilon_F$, hence, $\exp(\beta(\epsilon - \mu) + 1) = \exp(\beta(\epsilon - \epsilon_F) + 1) \xrightarrow[\beta \rightarrow \infty]{\epsilon < \epsilon_F} 1$ the exponential term becomes negligible. So the integration becomes,

$$N = \frac{2}{3} (2s + 1) \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \epsilon_F^{3/2} \quad (27)$$

from which we can write the expression for Fermi Energy $\epsilon_F = \frac{\hbar^2}{2m} \left(\frac{6\pi^2}{2s+1} \right)^{2/3} \left(\frac{N}{V} \right)^{2/3}$. that is, the Fermi energy is inversely proportional to the mass of the particles, and increases with particle density at power $2/3$.

Now the total energy of the such degenerate Fermi gas can also be calculated using the energy equation (22), but only up to $\epsilon = \epsilon_F$. We will get,

$$U = \frac{2}{5} (2s + 1) \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \epsilon_F^{5/2} \quad (28)$$

From eqn (26), we get the pressure of such a gas in this limit: $p = \frac{2}{5} \frac{N}{V} \epsilon_F = \frac{\hbar^2}{5m} \left(\frac{6\pi^2}{2s+1} \right)^{2/3} \left(\frac{N}{V} \right)^{5/3}$. This is a particularly interesting result. Even at zero temperature, the Fermi gas is associated with a certain finite pressure, and thus has to be kept inside a container. This direct consequence of Pauli's exclusion principle is related to the very stability of matter. This is often understood as Pauli (or Fermi) pressure. This pressure is not due to the collision of the electron gas with the walls of the container. Another characteristic variable of this kind of system is Fermi Temperature T_F defined as $T_F = \epsilon_F/k$.

The theory of the ideal Fermi gas has many applications of physical interest. One of the best examples is the description of thermal and transport properties of metallic compounds. According to the ideas of Drude and Lorentz, transport properties of metals may be explained by a model of (classical) free electrons. In the atoms of a metal, the outer electronic shell may liberate electrons from

the nuclei to form, at a first approximation, an ideal gas inside the volume of the metallic sample. Although certain conduction properties can indeed be qualitatively described by the classical treatment of Drude and Lorenz, the specific heat of metals changes with temperature (and thus violates the classical equipartition theorem). Besides the metallic systems, there are other physical systems, as nuclear matter or the ionized gas in the interior of some stars, that may be explained by a model of free fermions.

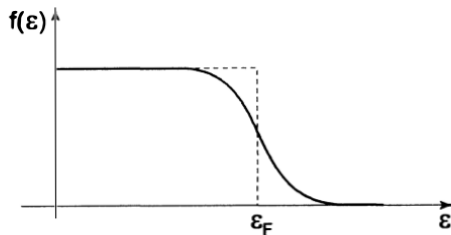


FIGURE 9.3. Fermi-Dirac distribution in the limit $T \ll T_F$.

4 Degenerate Fermi Gas $T \ll T_F$

In the previous section we were interested in completely degenerate Fermi gas which is close to absolute zero temperature $T \rightarrow 0$ (or $\beta \rightarrow \infty$). But most physical situations are not close to zero temperature, but some finite, yet small temperature. In many cases of physical interest, room temperature is much lower than the Fermi temperatures. If we assume the approximate model of an ideal gas to describe conduction electrons in copper, the Fermi temperature is of the order of 8×10^4 K. It is then relevant to make an attempt to obtain analytical results $T \ll T_F$. In the limit $T \ll T_F$, the Fermi-Dirac distribution, as sketched in figure above has the shape of a slightly rounded step compared to the previous plot at $T=0$. As a function of T , V , and N , the chemical potential μ should be slightly smaller than its value, ϵ_F , at zero temperature. We can say that some electrons that were occupying states below the Fermi energy have been excited to single-particle states above ϵ_F . This change takes place within a small band of energies, of order kT , in the neighborhood of ϵ_F . Therefore, the total number of excited electrons will be given by approximately,

$$\Delta N \approx (2s + 1) \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \sqrt{\epsilon_F} kT \quad (29)$$

Hence, the total change of energy of these electrons can be written as

$$\Delta U \approx kT \Delta N \approx (2s + 1) \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \sqrt{\epsilon_F} (kT)^2 \quad (30)$$

Thus, we have the specific heat at constant volume,

$$c_V = \frac{1}{N} \left(\frac{\partial U}{\partial T} \right)_{V,N} \approx 2(2s+1) \frac{V}{N} \frac{1}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \sqrt{\epsilon_F} V (k^2) T \quad (31)$$

Using eqn (27) and putting the expression of N here and writing $\epsilon_F = k T_F$ we can write specific heat at constant volume in terms of Fermi temperature, $c_V \approx \frac{3kT}{T_F}$. which is completely different from the constant value, $c_V = 3k/2$, for a classical ideal gas !! This linear dependence of the specific heat on temperature, which we have obtained from qualitative arguments, is indeed observed experimentally. However, at room temperature, since it is overwhelmed by the contribution of the elastic degrees of freedom (phonons), the electronic contribution to the specific heat of a metallic crystal is very small.

4.1 Sommerfeld Expansion

We now introduce an expansion proposed by Sommerfeld to obtain an asymptotic form for the specific heat at constant volume of the gas of free fermions. Note the integrals are of the form of

$$I(\epsilon) = \int_0^\infty f(\epsilon)\phi(\epsilon)d\epsilon \quad (32)$$

where $f(\epsilon)$ is the FD distribution function and $\phi(\epsilon)$ is the density of states. Note that all the equations like 22 or 23 are of this form. In general, we can take $\phi(\epsilon) = A\epsilon^n$ where A and n are +ve constants and $n > 1/2$. As $f(\epsilon)$ has the approximate shape of a step function (see FD distribution function curve), the derivative $f'(\epsilon)$ displays a rather pronounced peak at $\epsilon = \mu$. Performing integration by parts, we get

$$I(\epsilon) = f(\epsilon)\phi(\epsilon)|_0^\infty - \int_0^\infty f'(\epsilon)\psi(\epsilon)d\epsilon \quad (33)$$

where

$$\psi(\epsilon) = \int_0^\infty \phi(\epsilon')d\epsilon' \quad (34)$$

Taking into account that $f(\epsilon)$ vanishes exponentially as $\epsilon \rightarrow \infty$, we have

$$I(\epsilon) = - \int_0^\infty f'(\epsilon)\psi(\epsilon)d\epsilon \quad (35)$$

Since $f(\epsilon)$ displays a symmetric peak about $\epsilon = \mu$, we may write the expansion in Taylor form about μ

$$\psi(\epsilon) = \psi(\mu) + \left(\frac{\partial \psi}{\partial \epsilon} \right)_{\epsilon=\mu} (\epsilon - \mu) + \dots = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\partial^k \psi}{\partial \epsilon^k} \right)_{\epsilon=\mu} (\epsilon - \mu)^k \quad (36)$$

We thus have to calculate integrals of the form:

$$I_k = - \int_0^\infty f'(\epsilon)(\epsilon - \mu)^k d\epsilon = \frac{1}{\beta^k} \int_{-\beta\mu}^\infty \frac{e^x x^k}{(e^x + 1)^2} dx \quad (37)$$

for $k = 0, 1, 2, \dots$. At low temperatures ($T \ll T_F$), μ is close to ϵ_F and the lower limit of these integrals may be taken as $-\infty$. Indeed, owing to the form of the integrand, the error in these operations is of exponential order $\exp(-\beta\epsilon_F)$. Therefore, we have

$$I_k = -\frac{1}{\beta^k} \int_{-\infty}^\infty \frac{e^x x^k}{(e^x + 1)^2} dx + \mathcal{O}(\exp(-\beta\epsilon_F)) \quad (38)$$

For odd k , this integral vanishes due to odd function. For even k , except for exponential corrections, it is easy to show that $I_0 = 1$ and $I_2 = \pi^2/3\beta^2$. Thus, for $T \ll T_F$, we have the asymptotic result

$$I = \int_0^\mu \phi(\epsilon) d\epsilon + \frac{\pi^2}{6} (kT)^2 \left(\frac{\partial \psi}{\partial \epsilon} \right)_{\epsilon=\mu} + \dots \quad (39)$$

Now we use expansion of above to obtain asymptotic expressions for the internal energy

$$U = (2s+1)V \int_0^\infty f(\epsilon) C \epsilon^{3/2} d\epsilon = (2s+1) V C \left\{ \frac{2}{5} \mu^{5/2} + \frac{\pi^2}{4} (kT)^2 \mu^{1/2} + \dots \right\} \quad (40)$$

and the number of particles in the thermodynamic limit

$$N = (2s+1)V \int_0^\infty f(\epsilon) C \epsilon^{1/2} d\epsilon = (2s+1) V C \left\{ \frac{2}{3} \mu^{3/2} + \frac{\pi^2}{12} (kT)^2 \mu^{-1/2} + \dots \right\} \quad (41)$$

This final expression can be simplified a little bit using the expression of Fermi energy $\epsilon_F = \frac{\hbar^2}{2m} \left(\frac{6\pi^2}{2s+1} \right)^{2/3} \left(\frac{N}{V} \right)^{2/3}$,

$$\epsilon_F^{3/2} = \mu^{3/2} \left\{ 1 + \frac{\pi^2}{8} (kT/\mu)^2 + \dots \right\} \quad (42)$$

Homework: Work out the steps to establish the above relation from eqn 41 using the expression of ϵ_F mentioned above. **HINT:** replace N/V in eqn 41 in terms of ϵ_F .

From this equation, it is not difficult to write an asymptotic expression for the chemical potential as a function of temperature and density of particles N/V (which shows up through the expression of the Fermi temperature),

$$\mu = \epsilon_F \left\{ 1 - \frac{\pi^2}{12} (T/T_F)^2 + \dots \right\} \quad (43)$$

At low temperatures, we see that the chemical potential $\mu = \mu(T, V, N)$ is indeed smaller than ϵ_F . Inserting μ into eqn (40) for the internal energy, we have

$$U = \frac{3}{5}N\epsilon_F\left\{1 + \frac{5\pi^2}{12}(T/T_F)^2 + \dots\right\} \quad (44)$$

Therefore, the specific heat at constant volume is given by the asymptotic expression

$$c_V = \frac{\pi^2}{2}k\frac{T}{T_F} \quad (45)$$

which is the same linear relation as obtained previously as $c_V = 3kT/T_F$ with slightly different numerical prefactor.

5 Pauli Paramagnetism

Until now we only consider system of non-interacting fermions. Now we wish to consider fermions interacting with some external field, but not interacting with each other. In a system of free electrons, the magnetic field couples to the intrinsic (spin) and the orbital angular momenta. Let us consider these contributions separately. We refer to the interactions between an external magnetic field and the permanent magnetic moments, which leads to the phenomenon of paramagnetism. The Hamiltonian of the free electron gas in the presence of an external magnetic field is then given by

$$\mathcal{H} = \sum_{i=1}^N \left[\frac{1}{2m}p_i^2 - g\mu_B\vec{H}\cdot\vec{S}_i \right] \quad (46)$$

where \vec{S}_i is an operator of spin-1/2, $g = 2$ is the gyromagnetic factor, and μ_B is the Bohr Magneton. For a single electron, with the z-axis along the direction of the field, we have

$$\mathcal{H}_\infty = \frac{1}{2m}p_i^2 - g\mu_B H S_z \quad (47)$$

Therefore, the energy spectrum is $\epsilon_{k,\sigma} = \frac{\hbar^2 k^2}{2m} - \mu_B H \sigma$ with $\sigma = \pm 1$. The Grand canonical partition function can be written as

$$\ln \mathcal{Z} = \sum_k \sum_\sigma \ln \left\{ 1 + z \exp \left(-\frac{\beta \hbar^2 k^2}{2m} + \beta \mu_B H \sigma \right) \right\} \quad (48)$$

$$= \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int_0^\infty \epsilon^{1/2} \left\{ \sum_\sigma \ln [1 + z \exp(-\beta \epsilon + \beta \mu_B H \sigma)] \right\} d\epsilon \quad (49)$$

Thus, we can write,

$$\ln \mathcal{Z} = \ln \mathcal{Z}_+ + \ln \mathcal{Z}_-$$

where,

$$\ln \mathcal{Z}_\pm = \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int_0^\infty \epsilon^{1/2} \left\{ \ln [1 + z \exp(-\beta \epsilon \pm \beta \mu_B H)] \right\} d\epsilon$$

The average number of electron will be given by,

$$N = z \frac{\partial}{\partial z} \ln \mathcal{Z} = \langle N_+ + N_- \rangle$$

where,

$$\langle N_+ \rangle = \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int_0^\infty \epsilon^{1/2} \{ z^{-1} \exp(-\beta\epsilon \mp \beta\mu_B H) + 1 \}^{-1} d\epsilon \quad (50)$$

We can also use the same notation to write the expected value of the magnetization,

$$M = \mu_B \langle N_+ - N_- \rangle \quad (51)$$

5.1 Ground State Magnetization or low temperature Magnetization

From eqn 50 we have in the low temperature limit i.e. $\beta \rightarrow \infty$

$$\begin{aligned} \langle N_+ \rangle &= \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int_0^{\epsilon_F + \mu H} \epsilon^{1/2} d\epsilon = \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int_{-\mu_B H}^{\epsilon_F} (\epsilon - \mu_B H)^{1/2} d\epsilon \\ &= \frac{2}{3} \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} (\epsilon_F + \mu H)^{3/2} \end{aligned} \quad (52)$$

In a similar manner,

$$\begin{aligned} \langle N_- \rangle &= \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int_0^{\epsilon_F - \mu H} \epsilon^{1/2} d\epsilon = \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int_{\mu_B H}^{\epsilon_F} (\epsilon + \mu_B H)^{1/2} d\epsilon \\ &= \frac{2}{3} \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} (\epsilon_F - \mu H)^{3/2} \end{aligned} \quad (53)$$

where ϵ_F is the chemical potential at $T = 0$ (that is, the Fermi energy). To provide a pictorial understanding of this result, we say that the application of a magnetic field displaces the density of states for electrons with spin up ($\sigma = +1$), along the energy axis, by a quantity $-\mu_B H$; while the density of states for electrons with spin down is displaced in the opposite direction, by the same amount $+\mu_B H$. Therefore, we have the total number of electrons,

$$N = \langle N_+ + N_- \rangle = \frac{2}{3} \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \left((\epsilon_F + \mu H)^{3/2} + (\epsilon_F - \mu H)^{3/2} \right) \quad (54)$$

and the total magnetization is given by putting the same expression into eqn 51:

$$M = \mu_B \langle N_+ - N_- \rangle = 2 \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \mu_B \left((\epsilon_F + \mu H)^{3/2} - (\epsilon_F - \mu H)^{3/2} \right) \quad (55)$$

In the weak field limit i.e. $\mu_B H \ll \epsilon_F$ we write (by taking ϵ_F common and binomially expanding the rest)

$$N = \frac{4}{3} \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \epsilon_F^{3/2} + \mathcal{O} \left[\left(\frac{\mu_B H}{\epsilon_F} \right)^2 \right] \quad (56)$$

and the Magnetization in the weak field limit

$$M = 2 \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \mu_B \epsilon_F^{3/2} \left(\frac{\mu_B H}{\epsilon_F} \right) + \mathcal{O} \left[\left(\frac{\mu_B H}{\epsilon_F} \right)^3 \right] \quad (57)$$

Homework: Work out the steps to find the expressions in eqn 58 and 59 starting from eqn 56 and 57

The leading term of the Magnetization is given by

$$M = \frac{3}{2} N \mu_B \left(\frac{\mu_B H}{\epsilon_F} \right), \quad (58)$$

From this expression, we have the ground-state susceptibility in zero field,

$$\chi_0 = \left(\frac{\partial M}{\partial H} \right)_{T=0, H=0, V, N} = \frac{3N\mu_B^2}{2\epsilon_F} \quad (59)$$

which is one of the characteristic results of the Pauli paramagnetism.

6 Exercise

- 1) An ideal gas of fermions, with mass m and Fermi energy ϵ_F , is at zero temperature. The particles are moving with velocity \vec{v} . Find expressions for the expectation values $\langle v_x \rangle$ and $\langle v_x^2 \rangle$ where v_x is the x component of the velocity of a fermion.
- 2) Consider a gas of N free fermions which are moving on a surface of a metal sheet of surface area A at temperature T . Note that the motion of such fermions are 2 dimensional and also called 2D Fermion gas. Find the expression of specific heat of the system.