Module Theory Problem Set

For the students of
Integrated M.Sc. in Mathematics Semester X / M.Sc. in Mathematics Semester IV
Paper Code MA502/MT502 (Advanced Algebra)(SP)
Session 2019-20

Let R be a ring with 1 and M be a left R-module.

1. $\mathbb{Z}$-modules are same as abelian groups. Justify.
2. Show that a module over the ring $F[x]$ of polynomials in x with coefficients in the field F is the same as a vector space V together with a fixed linear transformation T of V (where the element x acts on V by the linear transformation T).
3. For any left ideal I of R define

$$IM = \{\sum_{finite} a_i m_i | a_i \in I, m_i \in M\}$$

to be the collection of all finite sums of elements of the form am where $a \in I$ and $m \in M$. Prove that IM is a submodule of M.

4. Show that the intersection of any nonempty collection of submodules of an R-module is a submodule.
5. If N is a submodule of M, the annihilator of N in R is defined to be $\{r \in R | rn = 0 \text{ for all } n \in N\}$. Prove that the annihilator of N in R is a 2-sided ideal of R.
6. If I is a right ideal of R, the annihilator of I in M is defined to be $\{m \in M | am = 0 \text{ for all } a \in I\}$. Prove that the annihilator of I in M is a submodule of M.
7. Let $n \in \mathbb{Z}^+, n > 1$ and let R be the ring of $n \times n$ matrices with entries from a field F. Let M be the set of $n \times n$ matrices with arbitrary elements of F in the first column and zeros elsewhere. Show that M is a submodule of R when R is considered as a left module over itself, but M is not a submodule of R when R is considered as a right R-module.
8. Suppose that A is a ring with identity 1_A that is a (unital) left R-module satisfying $r.(ab) = (r.a)b = a(r.b)$ for all $r \in R$ and $a, b \in A$. Prove that the map $f : R \to A$ defined by $f(r) = r.1_A$ is a ring homomorphism mapping 1_R to 1_A and that $f(R)$ is contained in the center of A. Conclude that A is an R-algebra and that the R-module structure on A induced by its algebra structure is precisely the original R-module structure.
9. Prove that $Hom_R(M, M)$ is an R-algebra if R is commutative.
10. Give an explicit example of a map from one R-module to another which is a group homomorphism but not an R-module homomorphism.

11. Let z be a fixed element of the center of R. Prove that the map $m \mapsto zm$ is an R-module homomorphism from M to itself. Show that for a commutative ring R the map from R to $End_R(M)$ given by $r \mapsto rI$ is a ring homomorphism (where I is the identity endomorphism).

12. Let $A_1, A_2, \cdots, A_n$ be R-modules and let B_i be a submodule of A_i for each $i = 1, 2, \cdots, n$. Prove that

$$(A_1 \times \cdots \times A_n)/(B_1 \times \cdots \times B_n) \simeq (A_1/B_1) \times \cdots \times (A_n/B_n).$$

13. Let I be a left ideal of R and let n be a positive integer. Prove that

$$R^n/IR^n \simeq R/IR \times \cdots \times R/IR \text{ (n times)}$$

where IR^n is defined as in Problem 3.

14. Assume R is commutative. Prove that $R^n \simeq R^m$ if and only if $n = m$, i.e., two free R-modules of finite rank are isomorphic if and only if they have the same rank.

15. What do you mean by an exact sequence and a short exact sequence? Give an example of a short exact sequence.

16. Let A, B and C be R-modules over some ring R. Then

    (a) the sequence $0 \to A \xrightarrow{\psi} B$ is exact (at A) if and only if ψ is injective.

    (b) the sequence $B \xrightarrow{\varphi} C \to)$ is exact (at C) if and only if φ is surjective.

17. State and prove The Short Five Lemma.