Special Theory of Relativity

Topics already covered : Gallilean Transformation and consequences, Postulates of Special theory of relativity, Principle of Simultanity, Length contraction, Time dialation, Twin paradox, Velocity addition theorem, Elastic and Inelastic collision, Variation of mass with velocity, Mass-energy equivalence, Relativistic energy and momentum

Topic 1: Minkowski Diagram

- In Minkowski Diagram, the time axis taken in the vertical direction, whereas the space axis is in the horizontal direction. In order to make identical dimension along both the axis, let us parametrize the vertical direction as $\omega = ct$, c being the velocity of light.

- Any event occuring at x at time t is represented on the $x - \omega$ plane by a point having co-ordinate (x, ω) and termed as 'space-time point' or 'world point'. The locus of 'world point' on the $x-\omega$ plane is reffered as a 'world line'.

- Let θ be the angle made by the tangent at any point on the world line with the ω axis, then $\tan\theta = \frac{dx}{d\omega} = \frac{dx}{dt} \cdot \frac{dt}{d\omega} = \frac{v}{c}$, v being the velocity of the particle. For a material particle, $v < c$ so that $\tan\theta < 1$ or $\theta < 45°$. So the world line of a light wave is a straight line inclined at an angle 45° to the ω axis.

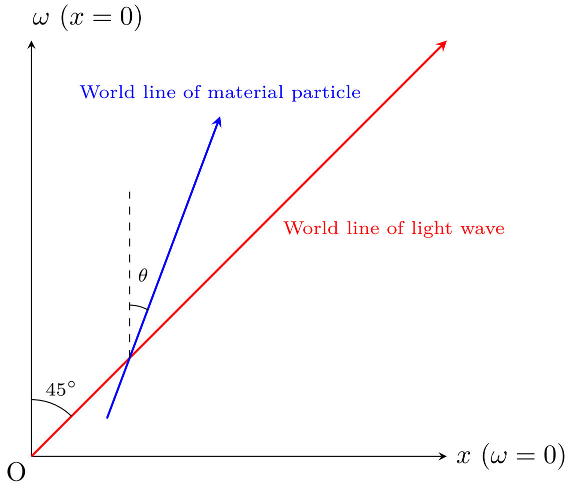

Figure 1: Minkowski diagram

- Consider any event $w.r.t$ S frame is represent by orthogonal $(x-\omega)$ co-ordinate system where $x = 0$ line represents the ω-axis and $\omega = 0$ line represents the x-axis. Now, let us assume a Lorentz transformation from S frame $(x - \omega)$ to S′ frame $(x' - \omega')$ *i.e.*

$$x' = \frac{x - \beta\omega}{\sqrt{1-\beta^2}}; \quad \omega' = \frac{\omega - \beta x}{\sqrt{1-\beta^2}} \tag{1}$$

Therefore, $x = \beta\omega$ line *i.e.* $x' = 0$ line will represent the ω' axis and $\omega = \beta x$ line *i.e.* $\omega' = 0$ line will represent the x' axis. It is therefore customary to conclude that Lorentz transformation of space-time involves transforming an orthogonal system to a non-orthogonal system.

- Consider $\omega' = 0$ (zero time) and $x' = 1$ (unit length) in S′ frame. Then

$$x - \beta\omega = \sqrt{1-\beta^2}; \quad \omega = \beta x \tag{2}$$

Solving these two equations, we find

$$\omega = \frac{\beta}{\sqrt{1-\beta^2}}; \quad x = \frac{1}{\sqrt{1-\beta^2}} \tag{3}$$

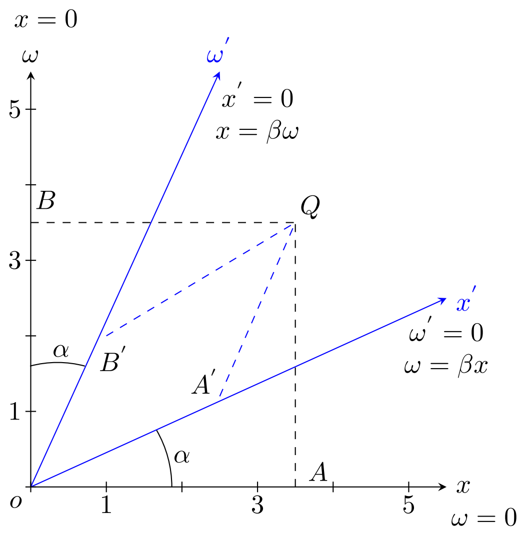

Figure 2: Minkowski diagram

Since $\beta < 1$, $x > 1$. This implies that the unit length interval along x′ is greater line segment than the unit length interval along x.

- Consider $\omega' = 1$ (unit time) and $x' = 0$ (zero length) in S′ frame. Then

$$x = \beta\omega; \quad \omega - \beta x = \sqrt{1-\beta^2} \tag{4}$$

Solving these two equations, we find

$$\omega = \frac{1}{\sqrt{1-\beta^2}}; \quad x = \frac{\beta}{\sqrt{1-\beta^2}} \tag{5}$$

Since $\beta < 1$, $\omega > 1$. This implies that the unit time interval along ω' is greater line segment than the unit time interval along ω.

- The space-time co-ordinates of a point (say Q) in the S and S′ frames can be obtained from the Minkowski diagram by drawing lines parallel to the axes from Q.

- Eliminating β between the equations(3) of ω and x, we obtain

$$x^2 - \omega^2 = 1 \qquad \texttt{Equation of a hyperbola} \tag{6}$$

Two branches $A_1B_1C_1$ and $A_2B_2C_2$ of this hyperbola, approach asympotically the 45 light ray world lines. For a particular value of β, the $x' - \omega'$ axes of S′ are drawn. If X′ is the point of intersection of the branch $A_1B_1C_1$ of the hyperbola $x^2 - \omega^2 = 1$ and x' axis, OX′ gives the unit length along x' axis.

- Eliminating β between the equations(5) of ω and x, we obtain

$$\omega^2 - x^2 = 1 \qquad \texttt{Equation of a hyperbola} \tag{7}$$

Two branches $A_3B_3C_3$ and $A_4B_4C_4$ of this hyperbola, approach asympotically the 45 light ray world lines. For a particular value of β, the $x' - \omega'$ axes of S′ are drawn. If T′ is the point of intersection of the branch $A_3B_3C_3$ of the hyperbola $\omega^2 - x^2 = 1$ and ω' axis, OT′ gives the unit time along ω' axis.

- Thus different points on the hyperbola $A_1B_1C_1$ give unit length in S′ for different values of β. Similarly different points on the hyperbola $A_3B_3C_3$ represents unit time in S′ for different values of β. Hence these hyperbolas are known as the calibration curves.

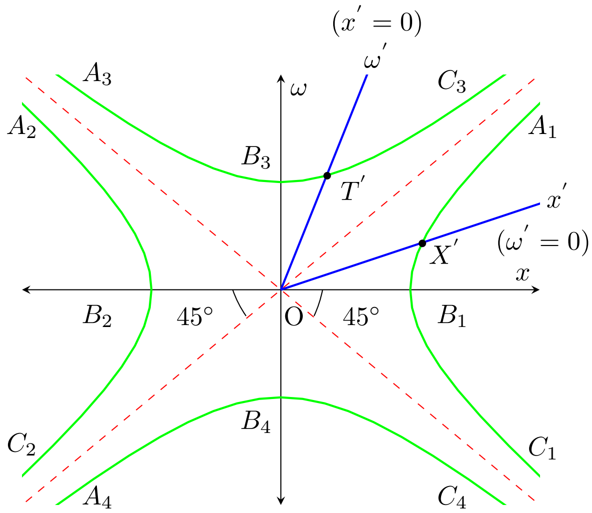

Figure 3: Hyperbolic calibration curves

Topic 2: Space-time intervals The space-time interval between two events in S frame occuring at (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2) is given by

$$s_{12}^2 = c^2 t_{12}^2 - l_{12}^2 \tag{8}$$

where, $t_{12} = t_2 - t_1$ and $l_{12} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$. If the same two events occured at (x_1', y_1', z_1', t_1') and (x_2', y_2', z_2', t_2') in S$'$ frame, then the space-time interval will be defined as

$$s'^2_{12} = c^2 t'^2_{12} - l'^2_{12} \tag{9}$$

where, $t'_{12} = t'_2 - t'_1$ and $l'_{12} = \sqrt{(x'_2 - x'_1)^2 + (y'_2 - y'_1)^2 + (z'_2 - z'_1)^2}$. Now if we consider S$'$ frame is moving *w.r.t.* S frame with a constant velocity then adopting Lorentz transformation, it can easily be shown that the space-time interval is invarient *i.e.* $s_{12} = s'_{12}$ so that

$$c^2 t_{12}^2 - l_{12}^2 = c^2 t'^2_{12} - l'^2_{12} \tag{10}$$

- If we assume that the two events to occur at the *same point* in S$'$ frame, we require $l'_{12} = 0$, then

$$s_{12}^2 = c^2 t_{12}^2 - l_{12}^2 = c^2 t'^2_{12} > 0 \tag{11}$$

  Consequently, a system of reference (S) with the required property exists if $s_{12}^2 > 0$ *i.e.* if the interval between the two events is a real number. The real intervals are said to be *time-like.*

- If we assume that the two events to occur at the *same time* in S$'$ frame, we require $t'_{12} = 0$, then

$$s_{12}^2 = c^2 t_{12}^2 - l_{12}^2 = -l'^2_{12} < 0 \tag{12}$$

  Consequently, a system of reference (S) with the required property exists if $s_{12}^2 < 0$ *i.e.* if the interval between the two events is a imaginary number. The imaginary intervals are said to be *space-like.*

- If we assume that the two events to occur at the *same point* and *same time* in S$'$ frame, we require $l'_{12} = 0$ and $t'_{12} = 0$, then

$$s_{12}^2 = c^2 t_{12}^2 - l_{12}^2 = 0 \tag{13}$$

  Consequently, a system of reference (S) with the required property exists if $s_{12}^2 = 0$ *i.e.* if the interval between the two events is zero. The zero intervals are said to be *light-like.*

Topic 3: Four-vector

- We know that in 3-D Cartesian co-ordinate, vector (or three-vector, say $\vec{A}$) is those physical entities components of which transforms as the co-ordinate transforms (translation or rotation). Likewise in 4-D Minkowski space, a four-vector is those physical entities components of which transforms as of the Lorentz co-ordinate transformation. If we assume $x_1 = x$, $x_2 = y$, $x_3 = z$ and $x_4 = ict$, then the Lorentz transformation can be recast as

$$x_1' = \gamma\left(x_1 + i\beta x_4\right); \quad x_2' = x_2; \quad x_3' = x_3; \quad x_4' = \gamma\left(x_4 - i\beta x_1\right) \tag{14}$$

Therefore, we can say four-vector (say $\{A_i\}$) is a set of four quantity which are transformed under the rules of Lorentz transformation, *i.e.* according to the rules,

$$A_1' = \gamma\left(A_1 + i\beta A_4\right); \quad A_2' = A_2; \quad A_3' = A_3; \quad A_4' = \gamma\left(A_4 - i\beta A_1\right) \tag{15}$$

Note that the set of four co-ordinates are $(x_i) = (x_1, x_2, x_3, x_4) = (x, y, z, ict)$ and thus with analogy, we can assume the presentation of a four vector as

$$\{A_i\} = \{A_1, A_2, A_3, A_4\} = \{A_x, A_y, A_z, ia\} = \{\vec{A}, ia\} \tag{16}$$

and after Lorentz transformation, the vector can thus be written as

$$\{A_i'\} = \{A_1', A_2', A_3', A_4'\} = \{\gamma\left(A_1 + i\beta A_4\right), A_2, A_3, \gamma\left(A_4 - i\beta A_1\right)\} \tag{17}$$

Now as by defination we have considered $A_4 = ia$,

$$\{A_i'\} = \{\gamma\left(A_1 - \beta a\right), A_2, A_3, i\gamma\left(a - \beta A_1\right)\} = \{\vec{A}', ia'\} \tag{18}$$

Let us assume two four-vectors $\{A_i\} = \{\vec{A}, ia\}$ and $\{B_i\} = \{\vec{B}, ib\}$ and two arbitrary scalars k_1 and k_2 such that

$$k_1\{A_i\} \pm k_2\{B_i\} = k_1\{\vec{A}, ia \pm k_2\{\vec{B}, ib\} = \{(k_1\vec{A} \pm k_2\vec{B}), i(k_1 a \pm k_2 b)\} = \{\vec{C}, ic\} = \{C_i\} \tag{19}$$

Therefore, we can construct a new four-vectors from two or more four-vectors by linear combinations with arbitrary scalars. It is also easy to show that the new four-vectors will follow the the requisite properties under Lorentz transformations.

- In 3-D cartesian coordinate, the magnitude of a vector (or three-vector) is preserved under rotation. In a similar fashion, the magnitude of a four-vector is preserved under Lorentz transformation in 4-D Minkowski space. In order to prove this, let us define the inner product between two four vectors $\{A_i\} = \{\vec{A}, ia\}$ and $\{B_i\} = \{\vec{B}, ib\}$ as

$$\{A_i\}.\{B_i\} = \vec{A}.\vec{B} - ab \tag{20}$$

Norm of a four-vector say $\{A_i\} = \{\vec{A}, ia\}$ is therefore can be written as

$$\sqrt{\{A_i\}.\{A_i\}} = \vec{A}.\vec{A} - a^2 \tag{21}$$

Let $\{A_i\} = \{\vec{A}, ia\}$ and $\{B_i\} = \{\vec{B}, ib\}$ be two four vectors. In the S' frame, the vectors looked upon as $\{A_i'\} = \{\vec{A}', ia'\}$ and $\{B_i'\} = \{\vec{B}', ib'\}$ so that the inner product

$$\{A_i'\} \cdot \{B_i'\} = A_1'B_1' + A_2'B_2' + A_3'B_3' - a'b' \tag{22}$$

Now using equation(18) and using the relation $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, it is easy to show that

$$\{A_i'\} \cdot \{B_i'\} = A_1B_1 + A_2B_2 + A_3B_3 - ab = \{A_i\} \cdot \{B_i\} \tag{23}$$

i.e. the inner product of two 4-vectors $\{A_i\}$ and $\{B_i\}$ in invarient underLorentz transformation. This conjecture further proves that the norm of a four-vector remains invarient under Lotentz transformation.

Four-displacement: If we assume $\{ds\}$ is the four-dispacement, then by defination

$$\{ds\} = \{d\vec{r}, icdt\} \tag{24}$$

so that

$$ds^2 = \{ds\}.\{ds\} = d\vec{r}.d\vec{r} - c^2(dt)^2 = dx^2 + dy^2 + dz^2 - c^2dt^2 \tag{25}$$

Four-velocity: As the interval between the two events under Lorentz transformation is invarient, we may write

$$(ds)^2 = (dx_1)^2 + (dx_2)^2 + (dx_3)^2 + (dx_4)^2 = (dx'_1)^2 + (dx'_2)^2 + (dx'_3)^2 + (dx'_4)^2 \tag{26}$$

where, $x_4 = ict$. If a clock is fixed in S' we have, $dx'_1 = dx'_2 = dx'_3 = 0$, so that

$$-c^2(dt')^2 = (dx_1)^2 + (dx_2)^2 + (dx_3)^2 - c^2(dt)^2 \tag{27}$$

As dt' is a proper time interval in S', we call $dt' = d\tau$. Therefore from (27),

$$-c^2(d\tau)^2 = (dt)^2\left[\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2 + \left(\frac{dx_3}{dt}\right)^2 - c^2\right] \tag{28}$$

$$\texttt{or,} \quad -c^2(d\tau)^2 = (dt)^2\left(v^2 - c^2\right)$$

$$\texttt{so,} \quad (d\tau)^2 = (dt)^2\left(\frac{v^2 - c^2}{c^2}\right) = (dt)^2\left(1 - \frac{v^2}{c^2}\right)$$

$$\texttt{or,} \quad (d\tau)^2 = \frac{(dt)^2}{\gamma^2} \qquad \texttt{where,} \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\texttt{or,} \quad d\tau = \frac{dt}{\gamma} \tag{29}$$

which is called time dialation. Again from (28), we have

$$-c^2(d\tau)^2 = (dt)^2\left[\left(\frac{dx_1}{d\tau}\frac{d\tau}{dt}\right)^2 + \left(\frac{dx_2}{d\tau}\frac{d\tau}{dt}\right)^2 + \left(\frac{dx_3}{d\tau}\frac{d\tau}{dt}\right)^2 - c^2\right]$$

$$\texttt{or,} \quad -c^2 = \left[\left(\frac{dx_1}{d\tau}\right)^2 + \left(\frac{dx_2}{d\tau}\right)^2 + \left(\frac{dx_3}{d\tau}\right)^2 - c^2\gamma^2\right] \quad \left[\texttt{from (23),}\ \gamma = \frac{dt}{d\tau}\right] \tag{30}$$

Now the four quantity $\frac{dx_1}{d\tau}, \frac{dx_2}{d\tau}, \frac{dx_3}{d\tau}$ and $i\gamma c$ all of these have dimension of velocity and are considered as the components of four-velocity *i.e.*

$$\{u_i\} = \left\{\frac{dx_1}{d\tau}, \frac{dx_2}{d\tau}, \frac{dx_3}{d\tau}, i\gamma c\right\} = \gamma\left\{\frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dx_3}{dt}, i\gamma c\right\} = \gamma\{v_x, v_y, v_z, ic\} = \gamma\{\vec{v}, ic\} \tag{31}$$

Equation (31) is the defination of four-velocity. Therefore,

$$\{u_i\}.\{u_i\} = \gamma^2\left(v^2 - c^2\right) = -c^2 \tag{32}$$

Four-acceleration: Four-acceleration $\{a_i\}$ is defined as

$$\{a_i\} = \left\{\frac{du_i}{d\tau}\right\} = \gamma\left\{\frac{d\vec{v}}{d\tau}, i\frac{dc}{d\tau}\right\} = \gamma\frac{d\vec{v}}{dt}\frac{dt}{d\tau} = \gamma^2\vec{a} \tag{33}$$

using equation, we find

$$\frac{d}{d\tau}\left[\{u_i\}\,.\,\{u_i\}\right] = \frac{d}{d\tau}(-c^2) = 0 \qquad \Rightarrow \{a_i\}.\{u_i\} = 0 \tag{34}$$

Therefore, four-velocity $\{u_i\}$ and four-acceleration $\{a_i\}$ are orthogonal to each other.

Four-momentum: Four-momentum can be evaluated by multiplying the four-velocity $\{u_i\}$ with the rest mass m_0 *i.e.*

$$\{p_i\} = m_0\,\{u_i\} = m_0\gamma\,\{\vec{v}, ic\} = \{m\vec{v}, imc\} = \left\{\vec{p}, \frac{iE}{c}\right\} \tag{35}$$

where $E = mc^2$ and $m = \frac{m_0}{\sqrt{1-\beta^2}} = \gamma m_0$. Moreover E is nolonger scaler in relativistic mechanics (It is not invarient under all inertial frame of reference.) and it is only a component of four-momentum. Now from equation(35) and using the rule of inner product between two four-vectors, we can write

$$m_0^2\{u_i\}.\{u_i\} = p^2 - \frac{E^2}{c^2} \tag{36}$$

Therefore using equation(32), the above equation becomes

$$E^2 = p^2c^2 + m_0^2c^4 \tag{37}$$

Four-force: Four-force $\{F_i\}$ is defined as

$$\{F_i\} = \left\{\frac{dp_i}{d\tau}\right\} = \left\{\frac{d\vec{p}}{dt}\frac{dt}{d\tau}, \frac{i}{c}\frac{dE}{dt}\frac{dt}{d\tau}\right\} = \gamma\left\{\vec{F}, \frac{i}{c}\frac{dE}{dt}\right\} = \gamma\left\{\vec{F}, \frac{i}{c}\vec{F}\cdot\vec{v}\right\}$$

Assignment

1. In an inertial frame S it is observed that a particle of mass 12 gm moves with a velocity of +50 cm/s along X-axis and approaches a second particle of mass 3 gm moving with a velocity of -20 cm/s along the same axis. It is also observed that after a head-on collision, the second particle acquires a velocity +20 cm/s along the X-axis. The same collision is also observed in another inertial frame S′ which has a velocity of +10 cm/s relative to S along the X-axis. Show that when momentum is conserved in S, it is also conserved in S′ under Galilean transformation.

2. An observer in the S-frame measures the area of a circle at rest in the X-Y plane to be 10 cm^2. To an observer in the S′ frame moving relative to the S-frame with a speed of 0.8c along the common X-axis, what will be the shape of the figure? Find also the area of figure measured by the S′ observer. [**Ans.** Ellipse, 6 cm^2]

3. Calculate the distance traversed by a particle during one mean life if it travels with a speed of 2.22×10^{10} cm/s. (Given proper mean life = 2.5×10^{-8} sec.) [**Ans.** 825.2 cm]

4. A rigid rod of length l_0 is moving with a velocity 0.8c in a direction at 30° to its own length. Find the length of the rod in motion and its inclination with X-axis in the S-frame. [**Ans.** $0.72 l_0$, 43.9°]

5. In S-frame at $t = 2 \times 10^{-3}$ s an explosion occurs at 4 km. For an S′ observer the explosion takes place at $x' = 30$ km.What is the time of the event for the S′ observer? [**Ans.** 2.002×10^{-3} s]

6. A space ship of rest length 120 m passes an observer on earth in 4.5 μs. Find its velocity relative to the earth. [**Ans.** 2.656×10^7 m/s]

7. Two signal lamps are lighted simulteously at points 1 km apart on a straight railroad track. How fast must a train move along the track in order that there is a time interval of 0.5 s between the events of lighting in the driver's frame ? Which lamp is lighted first in this frame ?

8. A rocket travels directly away from the earth with a velocity of 0.7c *w.r.t.* the earth. A missile, 5 m long, is launched from the rocket with a velocity of 0.8c relative to the rocket towards the earth.

   (a) Find the length of the missile as observed from the earth. [**Ans.** 4.869 m]

   (b) If the missile is launched 5 s after the rocket launch (according to the rocket clock), how long after the rocket launch will the missile hit the earth (according to an earth clock) ? [**Ans.** 28.59 s]

9. The explosion of a star at a distance of 10^{26} m creates neutrons of energy 8×10^6 J. Can any of these neutrons survive to arrive at the earth ? (given, half-life of a neutron = 10^3 s, rest mass of neutron = 1.67×10^{-27} kg.) [**Ans.** Yes, the neutrons will arrive at the earth]

10. In a frame S, the following two events occure:

$$\texttt{Event 1}: \ x_1 = x_0, \quad t_1 = x_0/c, \quad and \quad y_1 = z_1 = 0.$$
$$\texttt{Event 2}: \ x_2 = 2x_0, \quad t_2 = x_0/(2c), \ and \quad y_2 = z_2 = 0.$$

    Find the velocity of the frame S′ (*w.r.t.* S) in which these two events occur at the same time. What is the value of t in S′ at which these events are simultaneous ? [**Ans.** velocity =$-c/2$, time =$\sqrt{3}\frac{x_0}{2}$]

11. A body of rest mass M and moving with the velocity 0.8c collides head on with a stationary body of rest mass m. After the impact the two bodies stick together, and the combined body moves in the same direction with a velocity v. Find v and the rest mass of the combined body. [**Ans.** $v = \frac{4Mc}{5M+3m}$, rest mass =$\sqrt{\left(M + \frac{1}{3}m\right)(M+3m)}$]

12. A space traveller with velocity v synchronises his clock ($t' = 0$) with the earth friend ($t = 0$). The earth friend then observes both clock simultaneously, t directly and t' through a telescope. Show that when t' reads one hour, t reads $\sqrt{(1+\beta)/(1-\beta)}$ hour, where $\beta = v/c$.

13. (a) At what velocity does the total energy of a moving particle become exactly twice its rest energy ? [**Ans.** $\sqrt{3}$c/2]

    (b) In an accelerator, a particle of mass 1 GeV is accelerated to a total energy 5 GeV. What is the velocity of the particle in the rest frame of the accelerator ? [**Ans.** 0.98 c]

14. The measured life-time of a meson travelling with a speed of 2.8×10^{10} cm/s is 2.5×10^{-7} s in the laboratory. Calculate the proper life-time of the meson ? [**Ans.** 0.898×10^{-7} s]

15. Two particles move in opposite directions, each with speed 0.7c. What is the speed one particle as seen by the other ? [**Ans.** 0.94 c]

16. If 9 GeV electron collides headon with a 3.1 GeV positron which has the same mass. In this process both particles disappear and a new particle result. What is the mass and velocity of the new particle. [**Ans.** $\frac{10.6}{c^2}$, 0.49c]

17. A proton at rest of rest mass 938 MeV decays into kaon (494 Mev) and massless neutrino. What is the energy of kaon. [**Ans.** 599 Mev]

18. A particle x at rest decays to two particles a and b which fly off with equal and opposite momenta p in the rest frame of x. Show that

$$p = \frac{c}{2M_x}\left\{\left[M_x^2 - (m_a + m_b)^2\right]\left[M_x^2 - (m_a - m_b)^2\right]\right\}^{\frac{1}{2}}$$

    Where M_x,m_a and m_b are the rest masses of x, a, b repectively.

19. A pion at rest breaks up into a muon and a neutrino which fly off with equal and opposite momenta p. If m_π and m_μ are the rest masses of the pion and the muon respectively, then prove that

$$p = \frac{c\left(m_\pi^2 - m_\mu^2\right)}{2m_\pi}$$

20. The density of a stationary body is ρ_0. Find the velocity (relative to the body) of reference frame in which the density is 25% greater than ρ_0. [**Ans.** 1.34×10^8 m/s]

Reference

1. The Classical theory of Fields, L. D. Landau and E. M. Lifshitz, Butterworth Heinemann.
2. Introduction to Electrodynamics, D. J. Griffiths, Pearson India Education Services Pvt. Ltd.
3. An Introduction to the Special Theory of Relativity, R. Katz, D. Van Nostrand Company, Inc.
4. Special theory of Relativity, V. A. Ugarov, Mir Publishers.
5. The Special theory of Relativity, H. Dingle, John Wiley & Sons Inc.
6. Electricity and Magnetism, D. Chattopadhyay and P.C. Rakshit, Books and Allied (P) Ltd.