CLOSED GRAPH THEOREM

FOR THE STUDENTS OF INT. M.SC. IN MATHEMATICS, SEMESTER VIII AND
FOR PG STUDENTS OF MATHEMATICS, SEMESTER II
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1. Basic definitions and Results

Let (X, d_X) and (Y, d_Y) be two metric spaces.

Definition 1.1. (Graph of a mapping) If $T : X \to Y$ is a function, then the subset $G_T = \{(x, Tx) : x \in X\}$ of $X \times Y$ is called the graph of T.

Remark 1.2.
(1) If T is a linear operator and X, Y are vector spaces, then G_T is a subspace of $X \times Y$.
(2) Recall that if X and Y are normed spaces then $X \times Y$ is a normed space with $\|(x, y)\| = \|x\| + \|y\|$.
(3) The graph G_T is closed iff $x_n \to x$ and $Tx_n \to y \Longrightarrow y = Tx$

From the above remark it is clear that if T is a continuous function then its graph is closed. Therefore graph of every bounded linear operator from a normed linear space to a normed linear space is closed.

Definition 1.3. (Closed linear operator) Let X and Y be normed spaces and $T : X \to Y$ a linear operator. Then T is called a closed linear operator if its graph G_T is closed in the normed space $X \times Y$.

From the above discussion it is clear that all bounded linear operators on normed linear spaces are closed linear operator. Is the converse of the statement true?. The answer is No. Even discontinuous (unbounded) linear map can have a closed graph.

Example. Let $X = C^1[0, 1]$ and $Y = C[0, 1]$ where both have the sup norm $\|.\|_\infty$. Let $T : X \to Y$ be defined by $Tx = x^{'}$, the derivative of x. It is clear that T is linear. If $x_n(t) = t^n$, then $x^{'}_n(t) = nt^{n-1}$, and so $\|x_n\|_\infty = 1$, $\|Tx_n\|_\infty = n = n \|x_n\|_\infty$. Therefore $\|T\| \geq sup \|Tx_n\|_\infty = n$. Hence$T$ is unbounded, that is not continuous.
To show that G_T is closed, let $x_n \to x$ in X and $Tx_n \to y$ in Y. But the convergence with sup norm is uniform convergence. So, $x^{'}_n = Tx_n \to y$ uniformly. It follows that $x^{'} = y$, that is $y = Tx$. Thus the graph of T is closed.

Now the natural question is, when G_T closed implies T is bounded? Closed Graph Theorem answer this question.

Theorem 1.4. (The Closed Graph Theorem) *Let X and Y be two Banach spaces and $T : X \to Y$ be a linear operator. If the graph of T (i.e. G_T) is a closed subspace of $X \times Y$, then T is a bounded operator.*

Proof. Since G_T is a closed subspace of the Banach space $X \times Y$, it is also a Banach space. Let us consider $P_1 : G_T \to X$ defined by $(x, Tx) \to x$. Then P_1 is a linear operator from G_T onto X. Clearly, P_1 is injective and bounded as $\|P_1(x, Tx)\| = \|x\| \leq \|x\| + \|Tx\| = \|(x, Tx)\|$. Therefore open mapping theorem P_1 is a homeomorphism and hence $P_1^{-1} : X \to G_T$ is bounded. Again $P_2 : G_T \to Y$ defined by $P_2(x, Tx) = Tx$ is bounded. Hence $(T =)P_2 \circ P_1^{-1} : X \to Y$ defined by $x \to Tx$ is bounded as composition of two bounded/continuous function. This completes the proof. □

2. Some applications of the closed graph theorem

Application 2.1. Let $T : H \to H$ be an linear operator on a Hilbert space H such that $\langle Tx, y\rangle = x, Ty\rangle$ for all $x, y \in H$. Then T is bounded.

Proof. This follow immediately from the Closed Graph theorem. □

Application 2.2. Let X and Y be Banach spaces. Let Z be a subspace of X and $F : Z \to Y$be a linear map whose graph is closed in $X \times Y$. For $z \in Z$, let $\|z\|_F = (\|z\|^2 + \|F(z)\|^2)^{\frac{1}{2}}$. Show that Z is a Banach space and F is bounded in this norm. [Note that $\|.\|_F$ is the norm of the graph of F]

Proof. Clearly, $\|z\| \leq \|z\|_F$ and $\|F(z)\| \leq \|Z\|_F$ for all $z \in Z$. Let (z_n) be a Cauchy sequence in Z w.r.t. the norm $\|.\|_F$. Let $y_n = F(z_n)$. Then $\|z_n - z_m\| \leq \|z_n - z_m\|_F$, and $\|F(z_n) - F(z_m)\| = \|F(z_n - z_m)\| \leq \|z_n - z_m\|_F$. This shows that (z_n) and (y_n) are Cauchy sequences in X and Yrespectively. Since X and Y are Banach spaces, there exists $z \in X$ and $y \in Y$ such that $Z_n \to z$ and $y_n \to y$. So $(z_n, y_n) \to (z, y)$ in $X \times Y$, and hence (x, y) is in the graph of F, as it is closed. Hence $z \in Z$ and $y = F(z)$. Thus, $\|z_n - z\|_F^2 = \|z_n - z\|^2 + \|F(z_n - z)\|^2 = \|z_n - z\|^2 + \|y_n - y\|^2 \to 0$. Therefore, $z_n \to z$ in Z w.r.t. $\|.\|_F$. So Z is a Banach space. Since $\|F(z)\| \leq \|Z\|_F$, we see that F is continuous. □

Application 2.3. Let X, Y, Z be Banach spaces and $\{G_\alpha : \alpha \in A\}$ be a family of bounded linear maps from Y to Z. Suppose that if $G_\alpha(y) = 0$ for all $\alpha \in A$, then $y = 0$. If $F : X \to Y$ is linear and $G_\alpha \circ F \in B(X, Z)$ for every α in A, then $F \in B(X, Y)$.

Proof. It is enough if we can prove that the graph of F is closed.
Let $x_n \to x$ in X and $F(x_n) \to y$ in Y. Let $F(x_n) = y_n$. Then $y_n \to y$, and so $G_\alpha(y_n) \to G_\alpha(y)$ for all α.
But $G_\alpha(y_n) = G_\alpha(F(x_n)) = (G_\alpha \circ F)(x_n) \to (G_\alpha \circ F)(x)$.
Hence, $G_\alpha(y) = (G_\alpha \circ F)(x) = G_\alpha(F(x)) \Longrightarrow G_\alpha(y - F(x)) = 0$. Since this is true for all α, we must have $y = F(x)$. This completes the proof. □

Acknowledgments

This Class note is prepared from the books given to the references.

References

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